(c) If F_{ij} is an anti-symmetric tensor and the quantity S_{ijk} vanishes, where

$$S_{ijk} := \frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} (= 0)$$

Then show that the trivial solution for this would be the case that the tensor F_{ij} be of the form $F_{ij} := \partial_i A_j - \partial_j A_i$. Translate the above wisdom in the language of differential forms and in particular comment, on why the equation becomes merely an identity.

3+5+4=12

★★★

Bachelor of Science Examination, 2025

(3rd Year, 1st Semester)

PHYSICS

Paper : DSE/PHY/TH/01/A1

(Advanced Mathematical Methods)

Time : 3 *hours* *Full Marks* : 75

GROUP—A

Answer *any* **three** questions : 13×3=39

1. *(a)* Define conjugate elements and identify the equivalence classes.

(b) What is meant by Invariant (Normal) Subgroups, the Factor Group and Abelian Group? Discuss it.

(c) Clearly point out the condition for Reducibility and Decomposability of a group representation in terms of the invariant Subspaces. 4+4+5=13

2. Consider the symmetry transformations of a equilateral triangle and draw with its centre at the origin of x-y plane and parallel to x-axis, give rise to the D_3 symmetry group. One can make only six transformations *(a)* Identity; $D(e)$, *(b)* reflection about (1, 1′) line; $D(23)$, *(c)* reflection about (2, 2′) line; $D(31)$, *(d)* reflection about (3, 3′) line; $D(12)$, *(e)* anti-clockwise rotation about the center by about the center by $2\pi/3$; $D(321)$, *(f)* anti-clockwise rotation $4\pi/3$; $D(123)$.

(a) Construct the multiplication table.

(b) Find the two-dimensional and three-dimensional matrix realization of D_3 symmetry group.

(c) Find also the regular representation of D_3 symmetry group.

(d) Is there any one-dimensional representation of D_3 symmetry group?

(e) Construct the character table for all these representations. 13

3. *(a)* Successive translations about different axes do commute. This immediately leads to $[p_i, p_j] = 0$. Using translation operator establish the results.

(b) Write down the infinitesimal time evolution operator and then find out the operator for the case of finite time evolution.

(c) Show that finite rotation about different axes does not commute but infinitesimal rotation up to first order do commute. Using the result figure out the commutation relations among the generators. 4+4+5=13

(b) A geodesic on a given surface is a curve, lying completely on that surface and along which the distance between two points is the shortest. (For example, on the plane $\mathbb{R}^2$, a geodesic is the straight line). Now determine the equation of the geodesic on the surface of a sphere of radius R, that is for $S^2 : \{(x, y, z) : | x^2 + y^2 + z^2 = R^2\}$. 5+7=12

8. *(a)* Consider $\mathbb{R}^2$ and two points (x_1, y_1) and (x_2, y_2) on it. Suppose you define a surface of revolution by considering a curve joining these two points and then revolving it around the Y-axis. Find the curve for which the (surface) area of this surface of revolution is minimum.

(b) Let $w_1, w_2, w_3 \in \Omega^1(\mathbb{R}^3)$ be one-forms on $\mathbb{R}^3$, given by :

$$w_1 = x_2 dx_3 - x_3 dx_2$$
$$w_2 = x_3 dx_1 - x_1 dx_3$$
$$w_3 = x_1 dx_2 - x_2 dx_1$$

Then calculate : *(i)* $w_1 \wedge w_2$ and $w_1 \wedge w_2 \wedge w_3$. 8+4=12

9. *(a)* Expressing various vector quantities in the language of differential forms, write a note on the generalized form of the Stokes's theorem.

(b) Consider $\mathbb{R}^3$ and a generic 2-form $\Omega^2(x, y, z)$ living on it, i.e., of the type $\Omega^2 = a_1 dx \wedge dy + a_2 dy \wedge dz + a_3 dz \wedge dx$ with $a_i = a_i(x, y, z)$, evaluate $d\Omega$ and $d^2\Omega$.

4. *(a)* The rotation groups in the Euclidean plane, proper and improper, are defined as : $SO(2)/O(2) = A = 2 \times 2$ real matrix $A^T A = I_{2 \times 2}$, det $A = 1/\pm 1$. Show that $SO(2)$ is abelian, $O(2)$ nonabelian.

(b) Discuss the general form of representation and calculate the number of independent paramameters required, of the $SO(3)$ and $SU(2)$ groups. Discuss the connection between them. Give an example from Quantum mechanics in support of this connection.

(c) Let the polar and azimuthal angles that characterize $\hat{\mathbf{n}}$ be β and α respectively. You start with $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ the two-component spinor that represents the spin-up state. Given this, first rotate about the y-axis by an angle β and subsequently rotate by an angle α about the z-axis and obtain the final spin state in terms of the two-component spinor. 4+4+5=13

GROUP—B

Answer *any* **three** questions : 12×3=36

5. *(a)* Using the ideas from Calculus of Variations, formulate the variation of the functional :

$$\mathcal{I} = \int_i^f dx\, F(y, y', x), \text{ where } y = y(x),\ y' = \frac{dy}{dx}.$$

Hence obtain the Euler-Lagrange equation for the optimisation problem.

(b) Consider the centuries old *brachistochrone* problem of a rigid wire stretched between two poles of unequal vertical height and a certain horizontal distance apart. A light bead is to slide smoothly along the wire from end to end under the influence of gravity. What should be the shape of the wire so that the journey takes the minimum time. 5+7=12

6. *(a)* Consider the simple space $X = \mathbb{R}^4$. By first enumerating/listing the various kinds of 2-forms and 3-forms that can live on X, find the dimensions of the space of 2-forms and 3-forms on $\mathbb{R}^4$.

(b) Three of the most useful identities in vector calculus are the Green's theorem, Gauss' theorem and the Stokes' theorem and each of these can be re-written succinctly as rather trivial identities using the language of differential forms–show how? 7+5=12

7. *(a)* Now consider a special/shorter case of the problem of calculus of variations where the functional F does not depend on y. Thus consider the modified variational problem of optimizing.

$$\mathcal{I} = \int_i^f dx\, F(y', x), \text{ where } y = y(x),\ y' = \frac{dy}{dx}.$$

Obtain the special form of the Euler-Lagrange equation in this case and show that it can be integrated readily. (You need not start from first principles).