BACHELOR OF SCIENCE EXAMINATION, 2018

(1st Year, 1st Semester)

MATHEMATICS (Honours)

Paper - Core I

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

Notations and Symbols have their usual meanings.

Use a separate Answer-Script for each part.

Part - I (Algebra)

(15 marks)

1. Answer any *one*.

   (a) Define the gcd of two integers. Let a and b be two integers. Prove that there exist integers x and y such that gcd $\{a,b\} = ax + by$. Find integers x, y such that gcd $\{96, 162\} = 96x + 162y$. 5

   (b) Define a prime number. If 2^n+1 is an odd prime, prove that n is a power of 2. 5

[*Turn over*]

2. Answer any *one* :

(a) Prove that the equation $x^3 - 7x + 7 = 0$ has only real roots, one in (–4, –3) and two others in (1, 2). 5

(b) If $x = \log \tan\left(\frac{\pi}{4} + \frac{y}{2}\right)$, then prove that

$y = -i \text{ Log } \tan\left(\frac{\pi}{4} + i\frac{x}{2}\right)$, where x and y are real numbers. 5

3. Answer any *one* :

(a) Solve the following system of linear equations over the field of real numbers (if solutions exist) : 5

$$
\begin{aligned}
x_1 - 2x_2 + 0.x_3 + 2x_4 - 3x_5 &= 2 \\
2x_1 - 4x_2 + 2x_3 + 0.x_4 + 8x_5 &= 6 \\
x_1 - 2x_2 + 3x_3 - 3x_4 + 16x_5 &= 8
\end{aligned}
$$

(b) Let A be an $n \times n$ real matrix such that $I_n + A$ is invertible, where I_n is the $n \times n$ identity matrix. Define $A_* = (I_n - A)(I_n + A)^{-1}$. Show that $I_n + A_*$ is invertible and $(A_*)_* = A$. 5

[*Turn over*]

Part - II (Geometry)

(20 marks)

4. Answer any *five* questions : 4×5=20

(a) If d be the diameter to the circle passing through the pole and the points (r_1, θ_1), (r_2, θ_2), show that $d^2 \sin^2(\theta_1 - \theta_2) = r_1^2 + r_2^2 - 2\ r_1 r_2 \cos(\theta_1 - \theta_2)$.

(b) Prove that the two conics $\frac{l_1}{r} = 1 - e_1 \cos\theta$ and $\frac{l_2}{r} = 1 - e_2 \cos(\theta - \alpha)$ will touch one another if $l_1^2 (1 - e_2^2) + l_2^2 (1 - e_1^2) = 2 l_1 l_2\ (1 - e_1 e_2 \cos\alpha)$.

(c) Reduce the equation $4x^2 + 4xy + y^2 - 4x - 2y + a = 0$ to its canonical form and determine the type of conic represented by it for different values of a.

(d) Prove that the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ cuts the sphere $x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$ in a great circle, if $2(uu' + vv' + ww') = 2r'^2 + d + d'$, where r' is the radius of the second sphere.

(e) Find the equation of the cylinder whose generators are parallel to the line $x = \frac{y}{2} = \frac{z}{3}$ and guiding curve is the ellipse $x^2 + y^2 = 1, z = 3$.

(f) The section of a cone whose guiding curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ z = 0$ by the plane $x = 0$ is a rectangular hyperbola. Show that the locus of the vertex is $\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1$.

(g) Show that the generators of the hyperboloid $x^2 + y^2 - z^2 = 1$, which intersect on the plane $z = 0$ are at right angles.

Part - III (Calculus)

(15 marks)

Answer any *three* questions :

5. (a) Find the equation of the curve whose parametric equations are $x = \tan^{-1}(t),\ y = \left\{\tan^{-1}(t)\right\}^2\ \forall t \in R$.

[*Turn over*]

(b) If $y = e^{a \sin^{-1} x}$, then show that

$\left(1-n^2\right) y_2 - x y_1 - a^2 y = 0$ and hence

$\left(1-n^2\right) y_{n+2} - (2n+1) x\, y_{n+1} - \left(n^2 + a^2\right) y_n = 0$

Also find $y_n(0)$ when n is an even number. 1+4

6. (a) Define Envelope of a family of curves $F(x, y, \alpha) = 0$, where α is a parameter.

(b) Find the area between the cardiods $r = a(1+\cos\theta)$ and $r = a(1-\cos\theta)$. 1+4

7. (a) What do you mean by concavity of a curve w.r.t. a given point ?

(b) If $\mathrm{I}_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m x \sin nx \; dx \; (m, n \in Z_+)$, show that

$\mathrm{I}_{m,n} = \frac{1}{2^{m+1}}\left[2 + \frac{2^2}{2} + \frac{2^3}{3} + \frac{2^4}{4} + \ldots + \frac{2^m}{m}\right]$. 1+4

8. Define curvature of the curve at a given point. If ρ_1 and ρ_2 are radii of curvature at two points P and Q of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$, the tangents at which are at right angles then show that $\rho_1^2 + \rho_2^2 = 16a^2$. 1+4

[*Turn over*]

9. (a) Examine the asymptotes, if any, of the curve $y = xe^{\frac{1}{x}}$.

(b) Trace the curve $\left(a^2 + x^2\right)y = a^2 x$ with proper instification. 2+3