Ex/M.Sc/Math/22/4.4/B-2.30/2018

MASTER OF SCIENCE EXAMINATION, 2018

(2nd Year, 2nd Semester)

MATHEMATICS

Unit - 4.4 (B-2.30)

[Operator Theory-II]

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

Symbols/Notations have their usual meaning.

Answer any *five* questions.

1. (a) Let $T : H \to H$ be a bounded self adjoint linear operator on a complex Hilbert space *H*. Show that a scalar λ belongs to the resolvent set $\rho(T)$ iff there exists a constant $c > 0$ such that

$$\|(T - \lambda I)x\| \geq c\|x\|, \quad \forall x \in H.$$

(b) Show that the residual spectrum of a bounded self adjoint linear operator on a complex Hilbert space is empty. 6+4

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2. Let T be a bounded positive self adjoint linear operator on a complex Hilbert space H. Then show that there exists a unique bounded positive self adjoint linear operator A on H such that $A^2 = T$. 10

3. (i) Let P_1 and P_2 be two projections on a Hilbert space H. Then show that $P_1 - P_2$ is a projection if and only if $P_1(H) \subset P_2(H)$.

   (ii) Also show that $P_1 - P_2$ projects H onto Y where Y is the orthogonal complement of $P_1(H)$ in $P_2(H)$. 5+5

4. Let $T : H \to H$ be a bounded self adjoint linear operator on a complex Hilbert space H. For each real number λ, let E_λ be the orthogonal projection on the null space $N(T_\lambda^+)$, where T_λ^+ denotes the positive part of $T_\lambda = T - \lambda I$. Then show that $\varepsilon = (E_\lambda)$ is the spectral family on $[m, M]$ associated with T, where $m = \inf\{\langle Tx, x\rangle : \|x\| = 1\}$, $M = \sup\{\langle Tx, x\rangle : \|x\| = 1\}$. 10

5. Let $T : H \to H$ be a bounded self adjoint linear operator on a complex Hilbert space H and $\varepsilon = (E_\lambda)$ be the spectral family associated with T. Then a real scaler λ_0 is an

[*Turn over*]

eigenvalue of T iff the mapping $\lambda \to E_\lambda$ has a discontinuity at $\lambda = \lambda_0$. 10

6. Let $T : D(T) \to H$ be a linear self adjoint operator, not necessarily bounded, on $D(T) \subset H$ where H is a complex Hilbert space and $D(T)$ is dense in H. Then show that spectrum $\sigma(T)$ is real and closed. What happens to $\sigma(T)$ if T is bounded ? 8+2

7. (i) Let T be a bounded positive self adjoint linear operator on a complex Hilbert space H. Then show that for all $x \in H$

   $\|Tx\| \le \|T\|^{1/2} \langle Tx, x \rangle^{1/2}$.

   (ii) Let $T, S : H \to H$ be bounded linear operators on a complex Hilbert H and $S^* S \le T^* T$. If T is compact then show that S is compact. 5+5