**Ex/M.Sc./Math/12/2.3/133/2018**

# Master of Science Examination, 2018

**[1st Year, 2nd Semester]**

**MATHEMATICS**

**Unit - 2.3**

**(Functional Analysis)**

Full Marks : 50 Time : Two Hours

*The figures in the margin indicate full marks.*

Use a separate Answer-Script for each Part.

Unexplained Notations and Symbols have their usual meanings.

## Part - I

(25 Marks)

1. Justify with proper reasoning Q. No. (*f*) and any *three* from the rest. $2\times3+1=7$

   (a) Any real or complex Nector space can be equipped with a norm.

   (b) Inverse of any bijective bounded linear transformation between two normed linear spaces is also bounded.

[*Turn over*]

(c) Every norm on a real or complex vector space satisfies the parallelogram law.

(d) Let $N$ be an infinite dimensional normed linear space and $M$ be any nonzero normed linear space. Then there exists an unbounded linear transformation from $N$ to $M$.

(e) There may exist a normed linear space $N$ and a linear operator $T$ on $N$ such that $T$ has spectral value but has no eigenvalue.

(f) The geometric dual of any nonzero normed linear space is non zero.

2. Answer any *three* : 6×3=18

(a) (i) Supposing the Hahn Banach lemma for sublinear functional on real vector space prove it for complex vector space.

(ii) Give an example (with explanation) of a bounded linear operator $T$ on a normed linear space $N$ such that $T$ is not compact. 4+2

(b) (i) Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on a real or complex vector space $N$ such that $\left(N, \|\cdot\|_1\right)$ and

[*Turn over*]

$\left(N, \|\cdot\|_2\right)$ are Banach spaces. Prove that the following are equivalent :

(I) $\tau_{\|\cdot\|_1} = \tau_{\|\cdot\|_2}$

(II) $\tau_{\|\cdot\|_1} \subseteq \tau_{\|\cdot\|_2}$

(III) there exists a real number $k > 0$ such that $\| v \|_2 \leq k \| v \|_1$ for all $v \in N$.

(ii) Prove that for any normed linear space $N$, its geometric dual $N^*$ separates points. 4+2

(c) (i) Let $N$ be a normed linear space. Prove that $N$ is canonically embedded into its geometric double dual $N^{**}$.

(ii) Prove that a subset $X$ of a normed linear space $N$ is bounded if and only if $f\left(X\right)$ is a bounded subset of $\mathbb{R}$ or $\mathbb{C}$ for any $f \in N^*$. 4+2

(d) (i) Suppose $T : N \to M$ is an onto bounded linear transformation between normed linear spaces. Suppose there exists a positive real number $k$ such

[*Turn over*]

that $\| Tv \| \geq k\| v \|$ for all $v \in N$. Prove that $T^{-1}$ exists and is bounded.

(ii) Give an example of a non reflexive Banach space.

(iii) Give an example of an operator on a normed linear space having uncountably many eigenvalues. 2+2+2

(e) (i) Suppose $X$, $Y$ are Banach spaces. Suppose $\{Tn\}_{n=1}^{\infty}$ is a sequence of bounded linear transformations from $X$ to $Y$ such that $\lim_{n\to\infty} T_n x$ exists for all $x \in X$. Prove that the mapping $T : X \to Y$, given by $Tx = \lim_{n\to\alpha} T_n x$, is a bounded linear transformation.

(ii) Show that the spectrum of a bounded linear operator on a Banach space is bounded. 3+3

**Part - II**

(25 Marks)

Answer any *five* questions.

1. Define inner product space with an example. Prove that in an inner product space $V$, $|\langle x, y\rangle| \leq \| x \| \| y \|$ for all $x, y \in V$. 2+3

[*Turn over*]

2. "$l_p\ (p \neq 2)$ is a Banach space but not a Hilbert space". Justify the statement. What conclusion can you draw for $l_2$ space and why? 3+2

3. If $M$ is a closed subspace of a Hilbert space $H$, then prove that $H = M \oplus M^{\perp}$. 5

4. Let $H$ be a Hilbert space. Then prove that $f$ is a continuous linear functional on $H$ if and only if there exist a unique element $y \in H$ such that $f(x) = \langle x, y \rangle\ \forall x \in H$. Also prove that $\| f \| = \| y \|$. 5

5. If $\{e_i\}$ is an orthonormal sequence in a Hilbert space $H$ and $x \in H$, then prove that $\left( x - \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \right)$ is orthogonal to $e_j$, for each $j$. Hence prove that if $\{e_i\}$ is complete then $x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$. 3+2

6. (a) Prove that a continuous linear operator $T$ on a Hilbert space $H$ is selfadjoint if and only if $\langle Tx, x \rangle$ is real $\forall x \in H$.

(b) If $N$ is a normal operator on a Hilbert space $H$, then prove that $\| N^2 \| = \| N \|^2$. 3+2

7. Define projection operator $P$ on a Hilbert space. Prove that if $P$ is a projection operator and adjoint then $P$ is self adjoint. 2+3

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