MASTER OF SCIENCE EXAMINATION, 2018

[1st Year, 2nd Semester]

MATHEMATICS

Unit - 2.1

(Algebra - II)

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

Unexplained Notations and Symbols have their usual meanings.

Answer any *five* questions. 10×5=50

1. (a) Let F/K be a field extension. Define the degree of F/K. Find $[C:\mathbb{R}]$ and $\left[Q\left(\sqrt{2}\right):Q\right]$. Show that $[F:K]=1$ if and only if $K=F$. 1+1+1+2

   (b) Define algebraic field extension. Show that every finite field extension is an algebraic extension. Let F be a finite extension of K such that $[F:K]=p$ (prime). If $u \in F$ but $u \notin K$, then show that $F=K(u)$. 1+2+2

[*Turn over*]

2. (a) Determine a basis for $\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{2}\right)$ over Q. Hence show that $\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{2}\right)$ is a simple extension of $\mathbb{Q}$. 2+3

(b) Define splitting field of a polynomial over a field. Let $f(x) = x^4 + x^2 + 1 \in \mathbb{Q}[x]$. Find the splitting field S of $f(x)$ over $\mathbb{Q}$. Also find $[S:\mathbb{Q}]$. 2+2+1

3. (a) (i) Define algebraically closed field. Show that no finite field is algebraically closed.

(ii) Let F be a finite field of characteristic p. Show that $|F| = p^n$ for some positive integer n.

1+2+2

(b) (i) Define separable polynomial over a field. Show that every nonconstant polynomial over a finite field is separable.

(ii) Let F be a field of order 3^{100}. Find total number of subfields of F. 1+2+2

4. (a) (i) Show that $\sqrt[3]{2}$ is not constructible by straightedge and compass only. Hence show that it is impossible to duplicate the cube.

(ii) Show that regular pentagon is constructible by straightedge and compass only. 1+2+2

[*Turn over*]

(b) (i) Let F/K be a field extension such that $[F:K]=2$. Show that F is a normal extension of K.

(ii) Let $F=\mathbb{Q}\left(\sqrt[4]{2}\right)$ and $L=\mathbb{Q}\left(\sqrt[2]{2}\right)$. Show that F is a normal extension of L, L is a normal extension of $\mathbb{Q}$ but F is not a normal extension of $\mathbb{Q}$. 2+3

5. (a) Let $F=\mathbb{Q}\left(\sqrt{2},\sqrt{3}\right)$. Find the Galois group $G\left(F/\mathbb{Q}\right)$. Determine the order of the Galois group of $\mathbb{Q}(W)$ over $\mathbb{Q}$, where $W=e^{\frac{2\pi_i}{5}}$. 3+2

(b) When is a polynomial said to be solvable by radicals over a field F ? Show that the Galois group of the polynomial $f(x)=2x^5-10x+5$ over $\mathbb{Q}$ is S_5. Hence conclude that the equation $f(x)=0$ is not solvable by radicals. 1+3+1

6. (a) Let V be a finite dimensional vector space over a field F. Define idempotent and nilpotent operator on V. Show that an idempotent operators on V is diagonalizable but a nonzero nilpotent operator on V is not diagonalizable. 2+3

[*Turn over*]

(b) Let V be a finite dimensional vector space of dimension n over a field F and T be a linear operator on V. If T is a nilpotent operator of index K and $x \in V \setminus \{0\}$ is such that $T^{K-1}(x) \neq 0$ then show that $\{x, T(x), \ldots T^{K-1}(x)\}$ is a linearly independent subset of V. Hence show that T is a nilpotent operator of index n if and only if there is an ordered basis of V w.r.t. which the matrix of T is $\begin{pmatrix} 0 & 0 \\ I_{n-1} & 0 \end{pmatrix}$. 2+3

7. (a) Define elementary Jordan matrix and Jordan canonical form of a matrix. Does every square matrix over $\mathbb{R}$ have Jordan canonical form ? Justify your answer. 2+3

(b) (i) Let A be a 5×5 square matrix with characteristic polynomial $(x-1)^3 (x-2)^2$ and minimal polynomial $(x-1)^2 (x-2)$. Find the Jordan Canonical form of A.

(ii) Classify all possible Jordan canonical forms of a matrix $A_{5\times5}$ with characteristic polynomial $(x-\lambda)^5$. 2+3
