MASTER OF SCIENCE EXAMINATION, 2018

(1st Year, 1st Semester)

MATHEMATICS

Unit - 1.2

(Real Analysis)

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

Answer question no. 1 and any *three* from the rest.

1. Give an example of an uncountable set with Lebesgue measure zero. 2

2. (a) Define a Lebesgue measurable set. Show that $(a, \infty]$ is Lebesgue measurable. 1+4

   (b) Prove that following are equivalent

   (i) $E \subset R$ is Lebesgue measurable.

   (ii) For any $\in > 0$, there is an open set $O \supset E$ such that $\mu^*(O - E) < \in$.

[*Turn over*]

(iii) There is a G_δ-set $G \supset E$ such that

$$\mu^*(G-E)=0. \qquad 6$$

(c) If Δ is an open interval containing E, then show that

$$\mu(\Delta)=\mu^*(E)+\mu_*(\Delta \setminus E). \qquad 5$$

3. (a) If f and g are Lebesgue measurable functions then show that $f+g$ is also Lebesgue measurable. 3

(b) For a bounded measurable function f prove that there exists a sequence of simple functions $\{f_n\}_n$ such that $f_n \to f$ uniformly. 6

(c) For a ring R, prove that $M(R)=S(R)$, where $M(R)$ and $S(R)$ are the monotone class and σ-ring generated by R. 7

4. (a) Define an outer measure μ^*. For any $E \in H(R)$, show that

$$\mu^*(E)=\inf\{\overline{\mu}(F): E \subset F,\ F \in S(R)\}$$

$$=\inf\{\overline{\mu}(F): E \subset F,\ F \in \overline{S}\},$$

where symbols have their usual meanings. 1+5

[*Turn over*]

(b) In a set of finite Lebesgue measure E, prove that $f_n \to f$ a.e. implies $f_n \to f$ in (m) where f_n, f are all measurable functions on E. 5

(c) Define almost uniform convergence. If f_n, f are measurable and $f_n \to f$ a.u. then show that $f_n \to f$ a.e. and $f_n \to f$ in (m). 5

5. (a) For a bounded measurable function f defined on a measurable set E of finite Lebesgue measure prove that $\int_E f d\mu = 0$ iff $f = 0$ a.e. 4

(b) State and prove Monotone Convergence Theorem. 1+6

(c) Prove that f is a function of bounded variation on $[a, b]$ iff f can be expressed as the difference of two monotone increasing functions. 5

6. (a) If a bounded function f on $[a, b]$ is Riemann integrable on $[a, b]$ then show that f is Lebesgue integrable on $[a, b]$ and $R\int_a^b f = L\int_a^b f$. Is the converse true ? Justify your answer. 8+1

[*Turn over*]

(b) If f is Lebesgue integrable on $[a, b]$ and $F(x) = \int_a^x f \, d\mu$ for all $x \in [a, b]$ then prove that $F'(x)$ exists and $F' = f$ a.e. in $[a, b]$. 7