Ex/M.Sc/M/1.1/32/2018

MASTER OF SCIENCE EXAMINATION, 2018

(1st Year, 1st Semester)

MATHEMATICS

Unit - 1.1

(Algebra - I)

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

Use a separate Answer-Script for each part.

Unexplained symbols and notations have their usual meanings.

Part - I

(25 Marks)

Group - A

(25 Marks)

Answer any *five* questions.

1. State and prove Extended Cayley's Theorem of group. 2+3=5

[*Turn over*]

2. State Sylow's Second Theorem. By using this theorem show that H is a unique Sylow p-subgroup of a finite group G if and only if H is a normal subgroup of G. 2+3=5

3. Prove that every nilpotent group is solvable. Justify the converse of this result. 4+1=5

4. (a) Show that a group of order 96 has a normal sub-group of order 16 or 32.

   (b) Prove that every finite p-group is nilpotent. 3+2=5

5. Suppose G is a group of order pqr, where p, q, r are primes with $p > q > r$. Show that G is solvable. 5

6. Define group action. Let G be a finite group with $|G| = p^n$, where p is prime and $n \geq 1$. Show that $|Z(G)| > 1$ by using class equation. 2+3=5

7. (a) Does there exist an onto homomorphism from Z_6 onto Z_4 ? Justify your answer.

   (b) Define simple group. Show that no group of order 63 is simple. 2+1+2=5

[*Turn over*]

Part - II

(25 Marks)

Group - B

(15 Marks)

Answer any *three* questions. 5×3=15

8. (a) Let R be a commutative ring with identity such that every $x \in R$ satisfying $x^n = x$ for some $n \geq 2$. Show that every prime ideal of R is a maximal ideal of R.

   (b) Let R be a finite commutative ring with identity. Is every prime ideal of R a maximal ideal of R ? Justify your answer. 3+2=5

9. Show that every Euclidean Domain (ED) is a Principal Ideal Domain (PID). Is $\mathbb{Z}[x]$ a PID ? Justify your answer. 4+1=5

10. State Eisenstein criterion for irreducibility of a polynomial. Show that the cyclotomic polynomial

    $\varphi_p(x) = 1 + x + x^2 + \ldots + x^{p-1}$ is irreducible in $\mathbb{Z}[x]$, where p is a prime, by using Eisenstein criterion.

[*Turn over*]

11. Define Noetherian ring. Show that every principal ideal domain is a Noetherian ring. 2+3=5

12. State Hilbert basis theorem. Let R be a commutative ring with identity. Show that $R[x]/\langle x \rangle \simeq R$. Hence conclude that the converse of Hilbert basis theorem is also true. 1+3+1=5

Group - C

(10 Marks)

Answer any *two* questions. 5×2=10

13. Let R be a ring with identity. Let A and B be two submodules of a left R-module M. Show that

$$A/A \cap B \simeq (A+B)/B.$$ 5

14. (a) Let R be a ring with identity. Show that every finitely generated R-module is a homomorphic image of a finitely generated free R-module.

(b) Let R be a ring with identity having exactly 10 elements and M be a left R-module with exactly 20 elements. Is M a finitely generated free R-module ? Justify your answer. 3+2=5

[*Turn over*]

15. State fundamental structure theorem for finitely generated modules over a PID. With proper justification give an example of an infinite abelian group that is not a free $\mathbb{Z}$-module. 2+3=5