Ref. No.EX/SC/MATH/UG/CORE/TH/13/2024(S)

Bachelor of Science(Mathematics Hons), Supplementary Examination 2024

(3rd Year, 2nd Semester)

Group Theory-II

Full Marks : 40 **Time : 2 Hours**

The figures in the margin indicate full marks.

Special credit will be given for precise answer.

(Unexplained Symbols/ Notations have their usual meaning.)

Answer any four questions. $5 \times 4 = 20$

1.

(a) Suppose G is a group. Define the commutator subgroup G' of G. Prove that $G'$is a normal subgroup of G. Also prove that if H be a normal subgroup of G then G/H is Abelian if and only if $G' \subseteq H$. 7.

(b) Prove that S'_n is A_n 3

2.

(a) Determine all (upto isomorphism) Abelian groups of order 2000. 4

(b) Suppose G is a simple Abelian group. Prove that G is a finite group of prime order. 6

3.

(a) Prove that any group of order p^2q (p, q are distinct primes) is not simple. 6

(b) Give an example (with explanation) of an infinite p-group. Prove that the order of a finite p-group is some power of p. 2 + 2

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4.

(a) Suppose G is a finite group and H is a subgroup of G such that $o(G)$ does not divide $[G : H]!$. Prove that G is not simple. 6

(b) Suppose G is an abelian group having a non trivial element of order not equal to 2. Prove that $\mathrm{Aut}(G)$ has an element of order 2. 4

5.

(a) Which of the following can be the class equation of a group of order 10? Answer with reasons.
$10 = 1+1+1+2+5$; $10=1+1+2+2+2+2$; $10=1+3+2+2+2$. 3

(b) Suppose a finite group G has a congugacy class consisting of 2 elements. Prove that G has a subgroup of index 2 and hence prove that G is not simple. 3

(c) Find the invariant factors of the groups $G_1 = \mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{75}$, $G_2 = \mathbb{Z}_{12} \times \mathbb{Z}_{30} \times \mathbb{Z}_{15}$ and hence check if they are isomorphic. 4

6.

(a) Suppose G is a finite group with p a prime divisor of its order. Let n_p be the number of Sylow p-subgroups of G. Prove that for any Sylow p-subgroup P of G, $n_p = [G : N_G(P)]$ 4

(b) Using the fact: 'if a group G of order 60 has more than one Sylow 5-subgroup then G is simple', prove that A_5 is simple. 2

(c) Prove that a group G of order 105 has a unique Sylow 5-subgroup and a unique Sylow 7-subgroup. 4