5. Using Mean Value theorem of appropriate order, show that

$$x-\frac{x^3}{6} \le \sin x \le x-\frac{x^3}{6}+\frac{x^5}{120}$$ 5

6. *(a)* Evaluate $\lim\limits_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

(b) Find the value of a, b, c such that

$$\operatorname*{Lt}_{x\to 0}\frac{x(a+b-\cos x)-c\sin x}{x^5}=1$$ 2½+2½

★★★

Bachelor of Science Examination, 2024

(2nd Year, 1st Semester)

Mathematics

Paper : CORE – 5

(Theory of Real Function)

Time : *Two Hours* *Full Marks* : 40

Use separate Answer-Script for each Part.

Symbols / Notations have their usual meaning.

PART—I (20 Marks)

Answer ***any four*** questions : 5×4=20

1. *(a)* A function $f:\mathbb{R}\to\mathbb{R}$ satisfies the condition $f(x+y)=f(x)f(y)$ for all $x, y\in\mathbb{R}$. If f is continuous at $x = 0$, prove that f is continuous on $\mathbb{R}$. [3]

(b) Prove that $\lim\limits_{x\to 0}\sin\frac{1}{x}$ does not exist. [2]

2. *(a)* Let $f:\mathbb{R}\to\mathbb{R}$ be monotone on $\mathbb{R}$. Prove that the set of points of discontinuity of f is a countable set. [3]

(b) Give an example of a function f which is continuous on a closed interval I but $f(I)$ is not a closed interval. [2]

3. *(a)* Let $f:\mathbb{R}\to\mathbb{R}$ be continuous on $\mathbb{R}$. Prove that for every open subset G of $\mathbb{R}$, $f^{-1}(G)$ is open in $\mathbb{R}$. [3]

(b) Let $f:[0,\infty)\to[0,\infty)$ be a continuous function. If there exists some $M > 0$ such that $f(x)\le M$ for all $x\in[0,\infty)$, then prove that there is a point $c\in[0,\infty)$ such that $f(c) = c$. [2]

4. *(a)* Let I be a bounded interval and $f:I\to\mathbb{R}$ be uniformly continuous on I. Prove that f is bounded on I. [3]

(b) Prove or disprove : For any closed subset $A\subseteq\mathbb{R}$, there exists continuous function f on $\mathbb{R}$ which vanishes exactly on A. [2]

5. Let $f:\mathbb{R}\to\mathbb{R}$ be continuous and periodic of $\mathbb{R}$. Prove that f is uniformly continuous on $\mathbb{R}$. [5]

6. *(a)* Let $D\subseteq\mathbb{R}$ be a compact set and a function $f:D\to\mathbb{R}$ is continuous on D. Prove that f is uniformly continuous on D. [3]

(b) Prove or disprove : There exists a continuous map $f:\mathbb{R}\to\mathbb{R}$ such that $f(\mathbb{R})=\mathbb{Q}$. [2]

PART—II (20 Marks)

Answer ***any four*** questions : 5×4=20

1. Let a function $f(x)$ is defined by

$$f(x)=\begin{cases} x^p\sin\left(\dfrac{1}{x}\right) & , \quad x\neq 0 \\ 0 & , \quad x=0 \end{cases}$$

Find the value of p such that

(i) $f(x)$ is continuous at $x = 0$,

(ii) $f(x)$ is differentiable at $x = 0$. 5

2. Suppose $f:[a,b]\to\mathbb{R}$ be differentiable function. Let $a < c < d < b$. If $f'(c)<0$ and $f'(d)>0$, then show that there exists a point $\xi\in(c,d)$ such that $f'(\xi)=0$. 5

3. Examine the validity of Rolle's theorem for the function $f(x)=(x-a)^m(x-b)^n$; $\forall x\in[a,b]$ and m, n are (+)ve integers. 5

4. *(a)* If $f(x) = \tan x$, then show that

$$f^n(0)-{}^nc_2 f^{n-2}(0)+{}^nc_4 f^{n-4}(0)-\cdots\cdots=\sin\frac{n\pi}{2}$$

(b) If $\phi(x)$ is a polynomial in x and λ is a real number, then prove that $\exists$ a root of $\phi'(x)+\lambda\phi(x)=0$, between any pair of roots of $\phi(x)=0$. 2½+2½