5. *(a)* Give an example of a linear operator on *(i)* a finite dimensional vector space; *(ii)* an infinite dimensional vector space having no eigenvalues. Give reasons in support of your answers. 3

(b) Does there exist a linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ which maps $2x + y + 4z$ onto itself? If the answer is yes, is it unique? Answer with reason. 2

6. *(a)* Let V be the vector space of real polynomials with degree less than or equal to 2. Let D be the differentiation operator on V. Find the matrix A(say) of D with respect to the canonical basis of V. Also find the matrix B(say) of D with respect to the ordered basis $\{1, 1+2x, x+x^2\}$. Find a matrix P such that $B = P^{-1}AP$. 4

(b) Verify Cayley–Hamilton Theorem for the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_2(\mathbb{R})$. 1

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BACHELOR OF SCIENCE EXAMINATION, 2024

(Second Year, First Semester)

MATHEMATICS

PAPER : CORE – 6

(Ring Theory and Linear Algebra – I)

Full Marks : 40 *Time* : 2 *Hours*

Use separate Answer-Script for each Part.

Notations/Symbols have their usual meaning.

PART—I (20 Marks)

Answer ***any four*** questions : 5×4=20

1. When is a ring $(R, +, .)$ said to be commutative? Show that any ring of order 6 is commutative. Does there exist an integral domain of order 6? Justify. 1+2+2

2. Define Boolean ring. Does there exist a Boolean ring of order 10? Justify. Does there exist an infinite Boolean ring with zero-divisors? Justify. 1+2+2

3. *(i)* Is quotient ring of an integral domain always an integral domain? Justify.

(ii) Let R be an integral domain such that $AB = A \cap B$ for all ideals A, B of R. Show that R is a field. 2+3

4. *(i)* Let $R = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in \mathbb{R} \right\}$. Is R forms a field w.r.t. usual matrix addition and multiplication? Justify. Is R isomorphic to the field $\mathbb{R}$ of real numbers? Justify.

(ii) Are the rings $\mathbb{Z}_8$ and $\mathbb{Z}_4 \times \mathbb{Z}_2$ isomorphic? Justify.

(2+2)+1

5. Let R be a commutative ring with identity 1_R and R' be an integral domain with identity $1_{R'}$. If $f: R \to R'$ be a nonzero homomorphism then show that $f(1_R) = 1_{R'}$. Is R' an integral domain, essential? Justify. 3+2

6. Let R_1 and R_2 be two commutative rings with identity 1_R and $1_{R'}$ respectively. Let $f: R_1 \to R_2$ be a homomorphism and P be a prime ideal of R_2. Show that $f^{-1}(P) = \{a \in R_1 : f(a) \in P\}$ is a prime ideal of R_1. Is this result true, if P is a maximal ideal of R_2? Justify. 2+3

PART—II (Marks : 20)

(Linear Algebra)

Answer *Q. No. 3* & *Q. No. 4* and *any two* from the rest :

5×4=20

1. *(a)* Suppose $V_1 = \{(a,b,0) : a,b \in \mathbb{R}\}$, $V_2 = \{(0,a,b) : a,b \in \mathbb{R}\}$ and $V_3 = \{(0,b,0) : b \in \mathbb{R}\}$.

Which of V_i's ($i = 1, 2, 3$) pairwise form a direct sum? Give reasons in support of your answer. 2

(b) Let V be an $n(\geq 2)$ dimensional vector space over a field F and W be a $k(1 \leq k \lneqq n)$ dimensional subspace of V. Find a basis of the quotient space V/W. 2

(c) Prove that a change of basis matrix in a vector space is invertible. 1

2. *(a)* Let V be a vector space with dimension $2n$ (n is a positive integer). Prove that there exists a linear operator T on V with $\ker T = \operatorname{im} T$. 2

(b) Prove that for any linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ and any linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^2$, the linear transformation $T \circ S$ is neither 1 – 1 nor onto. Write the corresponding general result. 3

3. Suppose V is a finite dimensional vector space over a field F and $T : V \to V$ is a nonzero, non identity linear operator such that $T^2 = T$. Find a basis of V consisting of eigenvectors of T and find the matrix of T with respect to that basis. 5

4. *(a)* Prove that the algebraic multiplicity of an eigenvalue of a matrix is greater than or equal to its geometric multiplicity. 1

(b) Let A be a real matrix of order 50 with each entry 2. Without computing the characteristic polynomial find the eigenvalues of A. Also find their geometric and algebraic multiplicities. 4