

(4)

6. (a) Test the convergence of the following series :

$$\sum \frac{a^n}{n}, a > 0$$

- (b) Test the convergence of the following series :

$$\frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \dots \quad [3+2]$$

7. Let $\sum a_n$ be a convergent series of positive real numbers.
- (i) Prove that the sequence of n th partial sum of the series is bounded above.
- (ii) If $\{a_n\}$ is monotonically decreasing sequence, then prove that $na_n \rightarrow 0$ as $n \rightarrow \infty$. [2+3]

★ ★ ★

Ex/SC/MATH/UG/MAJOR/TH/11/101/2024

BACHELOR OF SCIENCE EXAMINATION, 2024

(1st Year, 1st Semester)

MATHEMATICS

PAPER : MAJOR – 101

(Real Analysis)

Time : 2 Hours

Full Marks : 40

Use a separate Answer-Script for each Part.

PART—I

(Marks : 20)

Answer *any five* questions : (4×5=20)

1. Show that the set $[0, 1]$ is uncountable. 4
2. Let F be an Archimedean ordered field. Show that if F satisfies least upper bound property then F has Cantor's nested interval property. 4
3. Show that interior of a set is the largest open set contained in the set. Find the derived set of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^c$. 2+2

(2)

4. Prove that in $\mathbb{R}$ finite intersection of open sets is open. Give an example to show that arbitrary intersection of open sets may not be open. 2+2
5. Find the closure of the set $\{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$. 4
6. Prove that every closed and bounded set in $\mathbb{R}$ is compact. 4
7. Show that an element x_0 is a limit point of a set S if and only if there exists a sequence $\{x_n\}$ of elements from $S \setminus \{x_0\}$ converging to x_0 . 4
8. Prove that the set $S = \{x \in \mathbb{Q} : 2 < x^2 < 3\}$ is both closed and open in $\mathbb{Q}$. Justify whether the set S is compact or not. 2+2

PART—II

(Marks : 20)

Answer **any four** questions :

1. (a) If the subsequences $\{x_{3n-2}\}, \{x_{3n-1}\}$ and $\{x_{3n}\}$ of a sequence $\{x_n\}$ converge to the same limit ℓ , then prove that $\{x_n\}$ converges to ℓ .
- (b) Prove that $n^{1/n} \rightarrow 1$ as $n \rightarrow \infty$ 3+2

MATH-322

[Continued]

(3)

2. Define nests of intervals. For any nest of closed intervals $\{[a_n, b_n]\}$, prove that there exists a unique real number x such that $x \in [a_n, b_n] \forall n$. 1+4
3. Prove that $\left\{ \left(1 + \frac{1}{n}\right)^n \right\}$ is convergent and $2 \leq \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \leq 3$. 5
4. (a) Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{7}$ and $u_{n+1} = \sqrt{7u_n} \forall n \geq 1$, converges to 7.
- (b) If $\{x_n\}$ is a bounded sequence and $\{y_n\}$ converges to 0, then prove that $\{x_n y_n\}$ converges to 0. 3+2
5. (a) Let $\sum u_n$ and $\sum v_n$ be two series of positive real numbers and $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \ell$. If $\ell \neq 0$, then prove that $\sum u_n$ and $\sum v_n$ converge or diverge together.
- (b) If $\sum u_n$ is a convergent series of positive real numbers, then prove that $\sum \frac{u_n}{s + u_n}$ is convergent for any non-zero real number $s > 0$. 3+2

MATH-322

[Turn Over]