Bachelor of Science Examination, 2024

(First Year, First Semester)

Mathematics (Honours)

(Old)

Paper : CORE - 02

(Group Theory - I)

Time : *Two Hours* *Full Marks* : 40

All questions carry equal marks

Answer ***any four*** *questions* 10×4=40

1. *(a)* Define a *group*. Let $\mathbb{Z}_n$ be the set of all integers modulo n, where n is a positive integer greater than 1. Show that $\mathbb{Z}_n$ is not a group under multiplication whereas $U_n = \{\vec{i} \in \mathbb{Z}_n \mid \gcd(n,i) = 1\}$ is a group under multiplication.

   (b) Let G be a group and $a, b \in G$ such that $a^2 = e$ and $ab^4a = b^7$, where e is the identity of G. Prove that $b^{33} = e$.

2. *(a)* Define a *subgroup* of a group. Let G be a group. Show that $C(a) = \{g \in G \mid ag = ga\}$ is a subgroup of G for all $a \in G$.

   (b) Prove that every infinite group has infinite number of subgroups.

3. *(a)* Let $\alpha, \beta, \gamma \in S_8$ such that $\alpha = (1\ 2\ 3\ 8)$, $\beta = (3\ 5\ 8)$ and $\gamma = (1\ 3)\ (4\ 7)$. Let $x = \alpha^3\beta^{-2}\gamma$. Find the order of x in S_8.

   (b) Define a *cyclic group*. Prove that every subgroup of a cyclic group is cyclic.

4. *(a)* Let G be group, H be a subgroup of G and $a, b \in G$. Prove that $aH = bH$ if and only if $a^{-1}b \in H$.

   (b) Define a *normal subgroup* of a group. Let H be a proper subgroup of a group G such that for all $x, y \in G \setminus H$, $xy \in H$. Prove that H is a normal subgroup of G.

5. *(a)* Define the *kernel* of a homomorphism (of groups). Let G be a group and H be a normal subgroup of G. Then show that there exists an onto homomorphism $f : G \to G / H$ such that kernel of f is H.

   (b) Show that the mapping $f : (\mathbb{Z} \times \mathbb{Z}, +) \to (\mathbb{Z}, +)$ defined by $f((a,b)) = a - b$ is a homomorphism, where $\mathbb{Z}$ is the set of all integers. Find the kernel of f.

6. *(a)* Show that $(\mathbb{Q}, +)$ and $(\mathbb{Q}, \cdot)$ are not isomorphic groups, where $\mathbb{Q}$ is the set of all rational numbers.

   (b) Show that every group is isomorphic to some subgroup of the group of all permutations of some set.

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