5. Discuss the bifurcation of the map $f_k(x) = k\tan^{-1}x$ at $k = 1$. Draw the corresponding bifurcation diagram. 5

6. Define the tent map $T(x)$ over the interval [0,1]. Let the map $G(x)$ defined in [0,1] is given by $G(x) = 4x(1-x)$. Prove that the Lyapunov exponent of an orbit of $T(x)$ and the Lyapunov exponent for the corresponding orbit of $G(x)$ are identical. 5

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Ex/SC/MATH/PG/CORE/TH/10/2024

MASTER OF SCIENCE EXAMINATION, 2024

(2nd Year, 1st Semester)

MATHEMATICS

PAPER : CORE - 10

(Dynamical Systems)

Time : *Two Hours* *Full Marks* : 40

Use a separate Answer-Script for each Part

Symbols/Notations have their usual meanings

PART—I

(Marks : 20)

Answer ***any two*** questions.

1. *(a)* Describe the stability around (0,0) and sketch phase portrait of the system $\dot{X} = AX$, where
$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},\ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ \alpha, \beta \in \mathbb{R}.$$

(b) Prove that any nondegenerate critical point of an analytic Hamiltonian system $\dot{x} = H_y(x, y)$, $\dot{y} = -H_x(x, y)$ is either a saddle or a center. Furthermore, (x_0, y_0) is a saddle iff it is a saddle of the function $H(x, y)$ and a strict local maximum or minimum of the function $H(x, y)$ is a center for $\dot{x} = H_y, \dot{y} = -H_x$. 5+5

2. *(a)* Characterise the bifurcation with respect to the parameter $\mu \in \mathbb{R}$ for the system

$$\dot{x} = -y$$

$$\dot{y} = \mu(1 - x^2)y - x$$

Draw the phase portrait.

(b) Find α-limit set, ω-limit set, limit cycle(s) and attractor of the system

$$\dot{r} = r(r-1)(r-2)$$

$$\dot{\theta} = -1$$

Sketch the phase portrait. 5+5

3. *(a)* Show that the system

$$\dot{x} = x(a - bx + cy)$$

$$\dot{y} = y(d - ey + fx)$$

$(a, b, c, d, e, f > 0)$ has no periodic orbit in the first quadrant.

(b) Determine Hamiltonian function of the system

$$\dot{\theta} = v$$

$$\dot{v} = -\sin\theta$$

(c) Consider the nonlinear autonomous system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ with an isolated equilibrium point $\bar{x}$. Define Lyapunov stability, asymptotic stability and global stability of the equilibrium point. 4+2+4

PART—II

(Marks : 20)

Answer ***any four*** questions.

1. When is a set of functions called linearly independent?

Prove that if $x_1(n), x_2(n), \ldots\ldots, x_k(n)$ be k linearly independent solutions of the difference equation.

$$x(n+k) + p_1(n)x(n+k-1) + p_2(n)x(n+k-2) + \ldots\ldots$$

$$p_k(n)x(n) = 0,\ n \geq n_0$$

then the general solution can be written as

$x(n) = c_1x_1(n) + c_2x_2(n) + \ldots\ldots\ldots\ c_kx_k(n).$ 5

2. Solve by annihilator method the difference equation

$y(n+2) - y(n) = n\cos(n\pi/2).$ 5

3. Suppose that for an equilibrium point x^*, $f'(x^*) = 1$ and $f''(x^*) = 0$. Prove that

(a) x^* is unstable if $f'''(x^*) > 0$ and

(b) x^* is asymptotically stable if $f'''(x^*) < 0$.

4. Find the general solution and hence discuss the stability of origin for the system of difference equation

$x(n+1) = Ax(n)$, where $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

Draw the phase portraits for various values of λ. 5