Master of Science Examination, 2024

(Second Year, First Semester)

Mathematics

Paper : DSE - 03

(A2 : Advanced Rings and Modules - I)

Time : *Two hours* | *Full Marks* : 40

Answer ***any four*** *questions* : 10×4=40

(Notations / Symbols have their usual meanings)

Let R be a commutative ring with identity 1.

1. *(a)* Define short exact sequence. If A and B are submodules of an R-modules M, then establish a short exact sequence

$$O \longrightarrow A \cap B \xrightarrow{\theta} A \times B \xrightarrow{\pi} A + B \longrightarrow O \cdot$$

2+3

(b) Show that $O \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow O$ is a split exact sequence of R-modules and R-homomorphisms if and only if $M \cong M' \oplus M''$. 5

2. *(a)* Show that every free R-module is projective. Is the converse true? Justify. 3+2

(b) Define divisible module. Let R be an integral domain. Show that every injective R-module is divisible. When is an integral domain said to be a Dedekind ring? 2+2+1

3. *(a)* Let A and B be two ideals of R. Show that $R/A \otimes R/B \cong R/A+B$. Hence deduce that $R/A \otimes R/B = 0$, if A and B are comaximal ideals of R. 4+1

(b) Define flat module. Give an example of a flat module. Is every R-module flat? Justify your answer. 2+1+2

4. *(a)* Let M be a maximal ideal of R such that $1 + m$ is a unit for all $m \in M$. Show that R is a local ring. Is this result true if M is not a maximal ideal of R? Justify. 3+2

(b) Let M be a maximal ideal of R. Show that R/M^n is a local ring for every positive integer n. Hence conclude that $\mathbb{Z}_{p^r}$ is a local ring for a prime p and $r \in \mathbb{N}$. 3+2

5. *(a)* Define ring of fractions. Let R be a Noetherian ring and S be a multiplicatively closed subset of R. Show that the ring of fractions $S^{-1}R$ is a Noetherian ring. 2+3

(b) Let S be a multiplicatively closed subset of R and $O \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow O$ be an exact sequence of R-modules and R-homomorphisms. Show that the following sequence

$$O \longrightarrow S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M'' \longrightarrow O$$

is an exact sequence of $S^{-1}R$-modules and $S^{-1}R$-homomorphisms. 3+2

6. *(a)* What are the primary ideals of the ring $\mathbb{Z}$? Show that a proper ideal P of R is primary if and only if every zero-divisor in R/P is nilpotent. 1+4

(b) Define radical ideal of ring. Show that a proper ideal I of R is a prime ideal of R if and only if I is radical and primary ideal of R. 1+4

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