Ref. No.: EX/SC/MATH/PG/DSE/TH/06/B16/2024(S)

M.Sc. Mathematics 2nd year 2nd Semester Special Supplementary Examination, 2024
Subject: Operator Algebra II
Paper: DSE-06 (B16)

Time: Two hours | Full Marks: 40

Symbols / Notations have their usual meanings.
Here, SOT and WOT denote Strong Operator Topology and Weak Operator Topology respectively.

Answer any four questions $(10 \times 4 = 40)$

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1. (a) Suppose T is an operator in a von Neumann algebra $\mathscr{A}$ acting on a Hilbert space H. Then prove that the orthogonal projection onto the closure of the range of T belongs to $\mathscr{A}$. [7]

   (b) Let H be a separable Hilbert space and $T \in B(H)$. Show that the norm $T \to \|T\|$ is not continuous in SOT. [3]

2. (a) Let H be a Hilbert space and $\mathscr{A}$ be a $*$-closed unital subalgebra of $B(H)$. Prove that $\mathscr{A}$ is SOT dense in $\mathscr{A}''$, where $\mathscr{A}''$ denotes the double commutant of $\mathscr{A}$. [7]

   (b) If H is an infinite dimensional separable Hilbert space, then prove that the closed unit of $B(H)$ is not compact in SOT. [3]

3. Let (Ω, F, μ) be a σ-finite measure space. Prove that $\{M_f : f \in L^\infty\}$ is a von Neumann algebra. [10]

4. (a) Let H be a Hilbert space and S be a subset of $B(H)$. Prove that the commutant of S is a WOT closed subspace of $B(H)$. [4]

   (b) Let H be a complex seperable Hilbert space. Let C be a unital C^* subalgebra of $B(H)$ and let $\mathscr{A}$ be the double commutant of C. Prove that the SOT closure of the set of self-adjoint operators in the unit ball of C is the set of self-adjoint elements in the unit ball of $\mathscr{A}$. [6]

5. (a) Suppose $\mathscr{A}$ is a von Neumann algebra and $\{P_\alpha\}$ is a family of projections in $\mathscr{A}$. Prove that $\wedge_\alpha P_\alpha$ and $\vee_\alpha P_\alpha$ both belong to $\mathscr{A}$. [5]

   (b) Prove that the function $f(t) = \frac{1}{1+t^2}$ is strongly continuous. [5]

[Turn over

6. (a) Let H be a Hilbert space and E be a spectral measure taking values in $B(H)$ on a measurable space (Ω, F). Prove that for any two vectors $u, v \in H$, $E_{u,v}$ defined by $E_{u,v}(D) = \langle u, E(D)v \rangle$, $D \in F$ is a complex measure on (Ω, F). [6]

   (b) Prove or disprove: Let H be a Hilbert space and $\{T_n\}$ be a sequence in $B(H)$. If $\{T_n\}$ converges to T is WOT, then for any fixed operators A, B in $B(H)$, the sequence $\{A\, T_n B\}$ converges to ATB in WOT. [4]