Ref. No.: EX/SC/MATH/PG/DSE/TH/03/A16/2024(S)

M.Sc. Mathematics 2nd year 1st Semester Special Supplementary Examination, 2024

Subject: Operator Algebra I

Paper: DSE-03 (A16)

Time: Two hours | Full Marks: 40

Notations: Here, the ground field for all algebras is the complex field $\mathbb{C}$ and $\sigma_{\mathscr{A}}(a), r_{\mathscr{A}}(a)$ denote the spectrum and spectral radius of the element a respectively, with respect to the Banach algebra / C^*-algebra $\mathscr{A}$.

Answer any four questions ($10 \times 4 = 40$)

1. (a) Let X be a Banach space and B(X) be the space of all bounded linear operators on X. Clearly $B(X)$ is a vector space. Show that $B(X)$ is a Banach algebra if the norm is defined by $||T|| = \sup\{||Tx|| : ||x|| \leq 1\}$ and if the multiplication is defined by composition.

   (b) Let $\mathscr{A}$ be a unital Banach algebra and $a \in \mathscr{A}$. Prove that $\sigma_{\mathscr{A}}(a) \neq \emptyset$. [4+6]

2. (a) If x is invertible in a Banach algebra $\mathscr{A}$, then prove that $\sigma_{\mathscr{A}}(x^{-1}) = \{\lambda^{-1} : \lambda \in \sigma_{\mathscr{A}}(x)\}$.

   (b) Let $\mathscr{A}$ be a unital Banach algebra, $\phi : \mathscr{A} \to \mathbb{C}$ be linear and $\mu_{\mathscr{A}}$ be the set of characters on $\mathscr{A}$. Prove that $\phi \in \mu_{\mathscr{A}}$, if and only if $\phi(1_{\mathscr{A}}) = 1$ and $\phi(a) \neq 0$ whenever a is invertible in $\mathscr{A}$. [3+7]

3. (a) Let $\mathscr{A}$ be a unital commutative Banach algebra and $\mu_{\mathscr{A}}$ be the set of characters on $\mathscr{A}$. Prove that the function sending a in $\mathscr{A}$ to $\hat{a}$ in $C(\mu_{\mathscr{A}})$ defined by $\hat{a}(\phi) = \phi(a)$, $\phi \in \mu_{\mathscr{A}}$ is a contractive Banach algebraic homomorphism from $\mathscr{A}$ to $C(\mu_{\mathscr{A}})$, where $C(\mu_{\mathscr{A}})$ denotes the set of continuous functions on $\mu_{\mathscr{A}}$.

   (b) Let $\mathscr{A}$ be a unital Banach algebra and $x, y \in \mathscr{A}$. Prove that $r_{\mathscr{A}}(xy) = r_{\mathscr{A}}(yx)$. [5+5]

4. (a) Let a be a normal element in a unital C^*-algebra $\mathscr{A}$ and $f : \sigma_{\mathscr{A}}(a) \to \mathbb{C}$ be any continuous function. Prove that $\sigma_{\mathscr{A}}(f(a)) = \{f(\lambda) : \lambda \in \sigma_{\mathscr{A}}(a)\}$.

   (b) Let a be a normal element in a unital C^*-algebra $\mathscr{A}$. Prove that a is self-adjoint if and only if $\sigma_{\mathscr{A}}(a) \subseteq \mathbb{R}$.

   (c) Suppose a is a normal element in a unital C^*-algebra $\mathscr{A}$ and $\sigma_{\mathscr{A}}(a)$ is contained in the imaginary axis of the complex plane. Show that $a^* = -a$ [3+4+3]

[Turn over

5. (a) If a is a normal element in a unital C^*-algebra $\mathscr{A}$ then prove that $r_{\mathscr{A}}(a) = ||a||$.

(b) Prove that for a unital C^*-algebra $\mathscr{A}$ there exists a pair (H, π), where H is a complex Hilbert space and $\pi : \mathscr{A} \to B(H)$ is an injective $*$-homomorphism.

[4+6]

6. (a) Let a be an element in a unital C^*-algebra $\mathscr{A}$ with identity $1_{\mathscr{A}}$. Prove that $||a||^2 \cdot 1_{\mathscr{A}} - a^*a$ is a positive element of $\mathscr{A}$.

(b) Suppose a is a normal element in a unital C^*-algebra $\mathscr{A}$. Prove that a is positive if and only if $\sigma_{\mathscr{A}}(a) \subseteq \mathbb{R}^+$. [4+6]