M. Sc. Mathematics Examination, 2024

(2nd Year, 1st Semester, Special Supplementary)

Mathematics

Paper – Core-10

[Dynamical Systems]

Time : 2 hours | Full Marks : 40

(Use separate answer script for each Part)

Part – I (Marks: 20)

Answer ***any two*** questions.

1. a) If $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, then show that

$$e^{At} = e^{at}\begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}.$$

b) What do you mean by Heteroclinic bifurcation?

c) State and prove fundamental theorem for linear system in $\mathbb{R}^n$. 3+2+5

2. a) Find α-limit set, ω-limit set, limit cycle(s) and attractor of the system

$$\dot{x} = x - y - x^2\left(x^2 + y^2\right)$$
$$\dot{y} = x + y - y^2\left(x^2 + y^2\right)$$

Draw the phase portrait.

[Turn over

b) Determine the nature of the equilibrium point (0, 0) for the system, $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

for $\alpha \in \mathbb{R}$. Draw the phase portrait. 5+5

3. a) State and prove Bendixson's criteria for the system $\dot{x} = f(x),\ x \in \mathbb{R}^2$.

b) Define Lyapunov function.

c) Find the evolution operator $\phi^t : \mathbb{R}^3 \to \mathbb{R}^3$ for the system $\dot{x} = f(x)$ with $f(x) = \begin{pmatrix} -x_1 \\ x_2 + x_1^2 \end{pmatrix}$ and hence show that the set $S = \left\{ (x_1, x_2) \in \mathbb{R}^2;\ x_2 = \frac{x_1^2}{3} \right\}$ is invariant under the flow ϕ^t. 3+2+5

Part – II (Marks: 20)

Answer ***any four*** questions.

1. Show that $\left\{ \cos\frac{n\pi}{2},\ \sin\frac{n\pi}{2} \right\}$ is a fundamental set of solutions of the equation $x(n+2) + x(n) = 0$.

2. Derive the expression for $F(n)$ from the Fibonacci series formula $F(n+2) = F(n+1) + F(n)$ starting from $F(1) = 1$ and $F(2) = 1$. Hence find the value of $\lim_{n\to\infty} \frac{F(n+1)}{F(n)}$.

3. Find solution of the system of equations

$$x(n+1) = 2x(n) - y(n)$$
$$y(n+1) = 4y(n)$$

Draw the flow trajectories in *xy*-plane starting from different initial conditions.

4. Find all fixed point(s) and period-two point(s) for the map $x_{n+1} = 2x_n^2 - 5x_n$. Also determine their stability.

5. Discuss bifurcation near $c = -1$ for the map $x_{n+1} = e^{x_n + c}$. Draw the bifurcation diagram.

6. Derive the formula for calculating Lyapunov exponent for a one-dimensional map $x_{n+1} = f(x_n)$. Hence find its value for the Tent map.