MASTER OF SCIENCE EXAMINATION, 2024

(Second Year, First Semester)

MATHEMATICS

PAPER : DSE - 04

(Operator Theory I)

Time : *Two Hours* *Full Marks* : 40

(Answer ***any four*** questions)

1. *(i)* Let T be a bounded linear operator on a complex Banach space $\mathbb{X}$ and $p(\lambda) = \alpha_n \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots \alpha_0$, $\alpha_n \neq 0$. Then show that $\sigma(p(T)) = p(\sigma(T))$.

   (ii) Let T be a bounded linear operator on a complex Banach space $\mathbb{X}$ and $p(\lambda) = \alpha_n \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots \alpha_0$, $\alpha_n \neq 0$. Show that the equation $p(T)x = y, x, y \in \mathbb{X}$, has a unique solution x for every $y \in \mathbb{X}$ iff $p(\lambda) \neq 0 \; \forall \lambda \in \sigma(T)$. 6+4

2. *(i)* Show that the resolvent set $\rho(T)$ of a bounded linear operator T on complex Banach space $\mathbb{X}$ is open.

   (ii) Show that the spectrum $\sigma(T)$ of a bounded linear operator T on a complex Banach space $\mathbb{X}$ is compact. 5+5

3. *(i)* Let (T_n) be a sequence of compact linear operators from a normed space $\mathbb{X}$ into a Banach space $\mathbb{Y}$. If (T_n) is uniformly operator convergent, say, $\| T_n - T \| \to 0$, then show that the limit operator T is compact.

(ii) Using the last result or otherwise, show that the operator $T : \ell^2 \to \ell^2$ defined by $Tx = y = (\eta_j), \eta_j = \frac{\varsigma_j}{2^j}$, is compact. 7+3

4. *(i)* Let $\mathbb{X}$ and $\mathbb{Y}$ be normed spaces and $T : \mathbb{X} \to \mathbb{Y}$ be a compact linear operator. Suppose that (x_n) in $\mathbb{X}$ is weakly convergent to x. Then show that the sequence Tx_n is strongly convergent to Tx in $\mathbb{Y}$.

(ii) Let T be a compact linear operator on an infinite dimensional space $\mathbb{X}$. Then show that 0 is a spectral value of T. 6+4

5. *(i)* Let T be a compact linear operator on a normed linear space $\mathbb{X}$ and $\lambda \neq 0$. Then for a given $g \in \mathbb{X}^*$, the equation $T^* f - \lambda f = g$ has a solution f iff $f(x) = 0$ for all x satisfying $Tx - \lambda x = 0$.

(ii) Show that the numerical range of a bounded linear operator on a Hilbert space is convex. 5+5

6. *(i)* Let T be a bounded linear operator on a complex Banach space $\mathbb{X}$. Then show that $r_\sigma(T) = \lim_{n\to\infty} \| T^n \|^{\frac{1}{n}}$.

(ii) Let $T : \ell^\infty \to \ell^\infty$ be defined by $x \to (\xi_2, \xi_3, \cdots)$, where x is given by $x = (\xi_1, \xi_2, \cdots)$. Show that $\lambda \in \rho(T)$ if $|\lambda| > 1$ and $\lambda \in \sigma(T)$ if $|\lambda| \leq 1$. 6+4

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