Master of Science Examination, 2024

(Second Year, First Semester)

Mathematics

Paper : DSE - 03

(Operator Algebra I)

Time : *Two Hours* *Full Marks* : 40

Notations : Here, the ground field for all algebras is the complex field $\mathbb{C}$ and $\sigma_{\mathscr{A}}(a)$, $r_{\mathscr{A}}(a)$ denote the spectrum and spectral radius of the element a respectively, with respect to the Banach algebra / C^*-algebra $\mathscr{A}$.

Answer ***any four*** questions (4×10=40)

1. Let $\mathscr{A}$ be a unital Banach algebra and $a \in \mathscr{A}$.

(a) Prove that $r_{\mathscr{A}}(a) = \lim_{n\to\infty} \| a^n \|^{\frac{1}{n}}$.

(b) Prove that if p is a polynomial with complex coefficients, then $\sigma_{\mathscr{A}}(p(a)) = p(\sigma_{\mathscr{A}}(a)) = \{p(\lambda) : \lambda \in \sigma_{\mathscr{A}}(a)\}$.

[7+3]

2. *(a)* Let $\mathscr{A}$ be a unital Banach algebra and $G(\mathscr{A}) = \{a \in \mathscr{A} : a$ is invertible in $\mathscr{A}\}$. Prove that the set $G(\mathscr{A})$ is open in $\mathscr{A}$.

(*b*) Let $\mathscr{A}$ be a commutative unital Banach algebra and $\mu_{\mathscr{A}}$ be the set of characters on $\mathscr{A}$. If $\phi \in \mu_{\mathscr{A}}$, then prove that $ker(\phi)$ is a maximal ideal of $\mathscr{A}$.

(*c*) Prove that every non-invertible element in a commutative unital Banach algebra $\mathscr{A}$ is contained in a maximal ideal of $\mathscr{A}$. [3+4+3]

3. (*a*) Let $\mathscr{A}$ be a unital commutative Banach algebra and $\mu_{\mathscr{A}}$ be the set of characters on $\mathscr{A}$. Prove that the Gelfand transform $\Gamma : \mathscr{A} \to C(\mu_{\mathscr{A}})$ is an isometry if and only if $\| x^2 \| = \| x \|^2$ for all $x \in \mathscr{A}$, where $C(\mu_{\mathscr{A}})$ denotes the set of continuous functions on $\mu_{\mathscr{A}}$.

(*b*) Let *a* be a self-adjoint element in a unital C^*-algebra $\mathscr{A}$. Prove that $\sigma_{\mathscr{A}}(a) \subseteq \mathbb{R}$ and $r_{\mathscr{A}}(a) = \|a\|$. [5+5]

4. (*a*) Let $\mathscr{A}$ be a unital C^*-algebra. If *a* is a positive element of $\mathscr{A}$, then prove that c^*ac is positive for all $c \in \mathscr{A}$.

(*b*) Let $\mathscr{A}$ be a unital C^*-algebra and ϕ be a state on $\mathscr{A}$. Prove that for any element $a \in \mathscr{A}$, $0 \le \phi(a^*a) \le \| a \|^2$.

(*c*) Prove that every element *a* in a unital C^*-algebra $\mathscr{A}$ is a linear combination of four unitary elements. [2+2+6]

5. (*a*) Let $\mathscr{A}$ be a unital C^*-algebra and ϕ be a state on $\mathscr{A}$. Prove that there exists a triple (H, π, u), where H is a Hilbert space, $\pi : \mathscr{A} \to B(H)$ is a unital *-homomorphism and $u \in H$ is a unit vector such that $\phi(a) = \langle u, \pi(a)u \rangle$ for all $a \in \mathscr{A}$.

(*b*) Let $\mathscr{A}$ be a unital C^*-algebra and ϕ be a positive linear functional on $\mathscr{A}$. Prove that $|\phi(a^*b)|^2 \le \phi(a^*a)\phi(b^*b)$ for all $a, b \in \mathscr{A}$. [8+2]

6. (*a*) Let *a* be a normal element in a unital C^*-algebra $\mathscr{A}$. Prove that *a* is a projection if and only if $\sigma_{\mathscr{A}}(a) \subseteq \{0, 1\}$.

(*b*) Let ϕ be a linear functional on a unital C^*-algebra $\mathscr{A}$. If $\phi(x)$ is real for every self-adjoint element *x* in $\mathscr{A}$, then prove that $\phi(a^*) = \overline{\phi(a)}$ for all $a \in \mathscr{A}$.

(*c*) Let *a* be an invertible element in a unital C^*-algebra $\mathscr{A}$. Prove that there exists a unique unitary $u \in \mathscr{A}$ such that $a = u|a|$, where $|a| = (a^*a)^{\frac{1}{2}}$. [3+2+5]

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