Master of Science Examination, 2024

(1st Year, 1st Semester)

Mathematics

Paper : Core - 02 (Real Analysis)

(Notations have their usual meanings)

Time : 2 *Hours* *Full Marks* : 40

Answer question No. 1 and any four from the rest.

1. *(a)* If A has finite Lebesgue measure, then show that there is a measurable set B with $\mu(B) = \mu(A)/2$.

(b) Prove that the Lebesgue outer measure is translation invariant.

(c) Give an example of a sequence of measurable functions which converges almost everywhere but does not converge in measure. 3+2+3

2. *(a)* For any finite interval I, prove that $\mu^*(I) = l(I)$

(b) If E_1 and E_2 are Lebesgue measurable, then prove that $\mu(E_1 \cup E_2) + \mu(E_1 \cap E_2) = \mu(E_1) + \mu(E_2)$. 6+2

3. *(a)* Prove that the following are equivalent :

   (i) E is Lebesgue measurable.

   (ii) For $\varepsilon > 0$, there exists an open set O containing E, such that $\mu^*(O - E) < \varepsilon$.

   (iii) There is a G_δ − set G containing E, such that $\mu^*(G - E) = 0$.

   (b) For any non-negative bounded measurable function f, show that there exists a monotone increasing sequence of simple functions which converges to f uniformly. 5+3

4. State and prove Egoroff's theorem. 8

5. *(a)* Prove that a bounded function defined on a measurable set with finite measure is Lebesgue integrable, iff it is measurable.

   (b) Give an example to show that a continuous function may not be a function of bounded variation. 6+2

6. *(a)* State and prove dominated convergence theorem.

   (b) If $\{f_n\}_n$ is a sequence of non-negative measurable functions defined on a measurable set E, then prove that

$$\int_E \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int_E f_n d\mu$$ 6+2

7. *(a)* If f is bounded and measurable on $[a,b]$ and $F(x) = \int_a^x f \, d\mu$, then show that F is differentiable almost everywhere in $[a, b]$ and $F'(x) = f(x)$ a.e. on $[a,b]$.

   (b) Show that every member of $S(E)$ can be covered by a countable collection of sets from E. If E is countable, is $S(E)$ countable? Answer with reasons. 6+2

8. *(a)* For any class of sets E, prove that the ring generated by E, $R(E)$ is countable, if E is countable.

   (b) Let μ be a finite measure defined on a ring R. Then prove that μ has an unique extension to a measure on $S(R)$. 5+3

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