Master of Science Examination, 2024

(1st Year, 1st Semester)

Mathematics

Paper : CORE – 01

(Abstract Algebra)

Time : 2 *Hours* *Full Marks* : 40

The figures in the margin indicate full marks.

(*Unexplained Symbols/Notations have their usual meanings.*)

Special credit will be given for precise answer.

Answer Q. No. **1** and *any* ***three*** from the rest : $10 \times 4 = 40$

1. Write **True** or **False** (with justifications) against each of the following statements :

(i) Every solvable group is nilpotent. 1

(ii) S_5 has a solvable subgroup of order 24. 1

(iii) A_6 has no subgroup of prime index. 1

(iv) The splitting field of $x^2 + 1 \in \mathbb{Q}[x]$ over $\mathbb{Q}$ is $\mathbb{C}$. 1

(v) The algebraic closure of $\mathbb{Q}$ is $\mathbb{C}$. 1

(vi) $\mathbb{Z}_2(t^2)$ is a perfect field where t is transcendental over $\mathbb{Z}_2$. 1

(vii) Every field has a proper algebraic extension. 1

(viii) Every field has a transcendental extension. 1

(ix) A field with 4 elements is a subfield of a field with 8 elements. 1

(x) Every ideal of $\mathbb{Z}[x]$ is finitely generated. 1

2. *(i)* Find a solvable series for the group S_4. 2

(ii) Prove that S_5 is not solvable. Use this to prove that S_n is not solvable for $n \geq 5$. 5

(iii) Prove that the angle 60º cannot be trisected using ruler and compass. 3

3. *(i)* Prove that every finite extension of a field is an algebraic extension. 2

(ii) Prove that there are irreducible polynomials over $\mathbb{Q}$ of arbitrary degree *n*. Use this to prove that the set of all algebraic numbers $\mathbb{A}$ is not a finite extension of $\mathbb{Q}$. 5

(iii) Let F/K be a field extension. Let $u \in F$ be algebraic over K. Find a basis of $K(u)$ over K. 3

4. *(i)* Prove that $\mathbb{Q}(\pi)$ and $\mathbb{Q}(e)$ are isomorphic as fields. Are they isomorphic as vector spaces over $\mathbb{Q}$? Justify. 5

(ii) Give an example (with explanation) of a *(a)* non-normal extension field and *(b)* non-separable extension field. 5

5. *(i)* Suppose *p* is a prime number and *n* is a positive integer. Prove that there exists a field with p^n elements. Also prove that any two fields with p^n elements are isomorphic. 5

(ii) Prove that a finite extension of a finite field is a Galois extension. 5

6. *(i)* Find the Galois group of *(a)* $x^4 - 10x^2 + 1$ over $\mathbb{Q}$ and *(b)* $x^3 - 7x + 1$ over $\mathbb{Q}$. 4+4=8

(ii) Suppose ζ is a primitive *p*th root of unity of $\mathbb{Q}$. Find the order of Gal $(\mathbb{Q}(\zeta)/\mathbb{Q})$. 2

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