MASTER OF SCIENCE EXAMINATION, 2025

(2nd Year, 1st Semester)

MATHEMATICS

PAPER : DSE-04 (A17)

(Operator Theory I)

Time : *Two Hours* *Full Marks* : 40

Attempt *any* **four** questions (10 × 4 = 40)

Notations *: Here, the space $B(X)$ denotes the space of all bounded linear operators from a normed linear space X to X.*

1. *(a)* Let X be a normed linear space and $A \in B(X)$. Prove that a scalar λ is an approximate eigenvalue of A if and only if there exists a sequence $\{x_n\}$ in X such that $\| x_n \| = 1$ for all $n \in \mathbb{N}$ and $\| A(x_n) - \lambda x_n \| \to 0$ as $n \to \infty$.

(b) Let $X = l^p$, where $1 \le p \le \infty$. If $T \in B(X)$ is defined by $(Tx)(j) = x(j+1)$ for $j = 1, 2, \ldots, x \in X$, find the spectrum of T. [4+6]=10

2. *(a)* Let X be a complex Banach space and $A \in B(X)$. If $\| A \| < 1$, then prove that $I - A$ is invertible and

$$\| (I - A)^{-1} \| \le \frac{1}{1 - \| A \|}.$$

(b) Let X be a normed linear space and $P \in B(X)$ such that $P^2 = P$. If $P \neq 0$, $P \neq I$, find the eigenvalues and spectrum of P. [5+5]=10

3. *(a)* Let A be a compact linear operator on a Banach space X. Prove that the set of eigenvalues of A is countable.

(b) Does there exist a compact linear map $T: l^\infty \to l^\infty$, which is surjective? Justify. [5+5]=10

4. *(a)* Let X, Y be Banach spaces and X^*, Y^* be the dual spaces of X and Y respectively. If $A: X \to Y$ is a compact linear operator, then prove that its adjoint operator $A^*: Y^* \to X^*$ is compact.

(b) Find the spectrum of a nilpotent operator A on a complex Banach space X. [7+3]=10

5. *(a)* Let $X = C[0,1]$ be a normed linear space with the sup-norm and k be a continuous scalar-valued function on the closed unit square

$S = \{(s,t) : 0 \leq s \leq 1, 0 \leq t \leq 1\}$.

Let $A: X \to X$ be defined by

$$A(x)(s) = \int_0^1 k(s,t)x(t)\,dt, \quad 0 \leq s \leq 1, x \in X$$

Show that A is compact.

(b) Let X be a complex Banach space, $A: X \to X$ be a compact linear operator and λ be a non-zero scalar. Prove that the equation $Ax - \lambda x = y$ has a solution for every $y \in X$ if and only if the homogeneous equation $Ax - \lambda x = 0$ has only the trivial solution $x = 0$.

[7+3]=10

6. Let H be a complex separable Hilbert space, A be a bounded linear operator on H and $w(A)$ be the numerical radius of A.

(a) Prove that the spectrum of A is contained in the closure of its numerical range.

(b) Prove that $w(A) \leq \|A\| \leq 2w(A)$. [5+5]=10

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