Bachelor in Computer Science and Engineering
2^{nd} Year, 2^{nd} Semester, 2024-2025
Semester Examination
Graph Theory and Combinatorics

Full Marks : 100 Time : 3 Hrs

Write answers to the point. Make and state all the assumptions(wherever made).
ALL PARTS OF A QUESTION SHOULD BE ANSWERED TOGETHER

(Section A) Answer all question $(3 \times 10 = 30)$

(1) In any graph show that the number of odd vertices is even.

(2) Define strongly and weakly connected graph. Give examples.

(3) For each of the following, draw an Eulerian graph that satisfies the conditions, or prove that no such graph exists.

i. An even number of vertices, an even number of edges.

ii. An even number of vertices, an odd number of edges.

(4) Let $\mathbf{G(V,E)}$ be a graph of 20 vertices numbered 1 to 20. Two vertices (v_i, v_j) are connected only if $|i-j| = 8$ or $|i-j| = 10$. Find the number of components of the graph.

(5) Show that $K_{m,n}$ is regular if and only if $m = n$

(6) Six people are seated at a round table to play a game of cards.

i. Is the seating arrangement around the table a linear or circular permutation?

ii. How many possible seating arrangements are there?

(7) State the extended pigeonhole principle. How many people must have to guarantee that at least 9 of them will have birthday in the same day of the week?

(8) Solve the linear Diophantine equation: $7x - 9y = 3$.

(9) A coin is tossed until we obtain a sequence of an odd number of "heads", followed by one "tail", for the first time. How many possible sequences of n tosses are there?

(10) In how many ways can the vertices of a regular n-gon be coloured in such a way that no two adjacent vertices are coloured with the same colour?

(Section B) Answer any 5 questions $(5 \times 5 = 25)$

(1) What are components in a graph? Prove that a simple graph with n vertices and k components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges

(2) What is a connected graph? Let $\mathbf{G(V,E)}$ be a connected graph with equally n vertices and edges respectively. Show that $\mathbf{G(V,E)}$ has exactly one cycle.

(3) Show that every simple connected planar graph $\mathbf{G(V,E)}$ with less than 12 vertices must have a vertex of $degree(v) \leq 4$

(4) Let $n \geq 2$. Show that there is a tree with degree sequence $d_1, d_2, \ldots, d_n$, if and only if $d_i > 0$ for all i, and $\sum_{i=1}^{n} d_i = 2(n-1)$.

(5) Suppose a simple graph $\mathbf{G(V,E)}$ on n vertices has at least $\left(\frac{(n-1)(n-2)}{2} + 2\right)$ edges. Prove that $\mathbf{G(V,E)}$ has a Hamilton cycle.

(6) Suppose a connected graph $\mathbf{G(V,E)}$ has degree sequence $d_1, d_2, \ldots, d_n$. How many edges must be added to $\mathbf{G(V,E)}$ so that the resulting graph has an Euler circuit? Explain.

(Section C) Answer any 5 questions $(5 \times 5 = 25)$

(1) 25 boys and 25 girls sit around a table. Prove that it is always possible to find a person both of whose neighbours are girls

(2) Without using the inclusion-exclusion principle, find the number of ordered quadruples (x_1, x_2, x_3, x_4) of positive odd integers that satisfy $x_1 + x_2 + x_3 + x_4 = 98$.

(3) Use the principle of inclusion and exclusion to find the number of solutions of the equation $x_1+x_2+x_3 = 18$, where x_1, x_2, x_3 are non-negative numbers such that $x_1 \leq 5, x_2 \leq 6, x_3 \leq 18$.

(4) A particle, moving in the horizontal direction, traverses a distance in each second. This distance is equal to two times the distance it travelled in the previous second. If a_r denotes the particle in the r^{th} second, find a_r subjected to the initial conditions $a_0 = 3, a_3 = 10$

(5) Suppose a drawer contains three red beads, four blue beads, and five green beads. Use a generating function to determine the number of ways to select six beads if one must select at least one red bead, an odd number of blue beads, and an even number of green beads. Assume that beads of the same color are indistinguishable, and that the order of selection is irrelevant.

(6) Show that a De Bruijn sequence of order 3 is 00010111.

(Section D) Answer any 1 question (20)

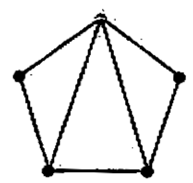

Figure 1: Graph

(1) Find the chromatic number of the graph Figure 1 by using ~~the~~ a designed algorithm, ~~in this section~~. Draw all of the graphs $G + e$ and G/e generated by the alorithm in a "tree structure" with the complete graphs at the bottom, label each complete graph with its chromatic number, then propogate the values up to the original graph. Also find the chromatic polynomial of the graph.

(2) Solve $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$ using generating function in terms of F_i. Also, show that as $\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2}$