Best coapproximation in Banach spaces: A Birkhoff-James orthogonality approach

Shamim Sohel
(Index No.: 36/22/Maths./27)

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FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS

JADAVPUR UNIVERSITY
Kolkata - 700 032, India
Telephone : 91 (33) 2457 2269

CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "**Best coapproximation in Banach spaces: A Birkhoff-James orthogonality**" submitted by **Shamim Sohel** who got his name registered on 15/02/2022 (**Index No.: 36/22/Maths./27**) for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon his own research work under the supervision of **Prof. Kallol Paul**, Department of Mathematics, Jadavpur University, Kolkata 700032, India and that neither this thesis nor any part of it has been submitted for any degree/diploma or any other academic award anywhere before.

K Paul 16.06.25
....
(Prof. Kallol Paul)
(Signature of the Supervisor
and date with official seal)

Prof. Kallol Paul
Professor (On Lien)
Department of Mathematics
Jadavpur University
Kolkata-700032, India

Dedicated to my parents

Mrs. Kamrunnesha Khatun

and

Late Sk. Diljan

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Shamim Sohel

Abstract

In this dissertation, we explore the best coapproximation problem, a notion complementary to the classical best approximation problem, within the framework of Banach space using the notion of Birkhoff-James orthogonality. We study the problem in the space of all $n \times n$ diagonal matrices from a computational perspective. We completely characterize the best coapproximation(s) to a given matrix T out of a given subspace $\mathbb{Y}$ of the space of diagonal matrices $\mathcal{D}_n$. As a consequence of our study, we solve the problem computationally in the space ℓ_∞^n. We present a tractable approach to solve the best coapproximation problem in the space ℓ_1^n, leading to a complete characterization of coproximinal and co-Chebyshev subspaces. We next introduce two types of subspaces which may be regarded as the least favorable situations from the existence of best coapproximations, named as anti-coproximinal subspaces and strongly anti-coproximinal subspaces. We obtain some necessary and sufficient conditions for strongly anti-coproximinal subspaces in general Banach spaces, separatelty. In particular, we show that strictly convex and smooth Banach spaces does not contain any strongly anti-coproximinal subspaces, and we provide a characterization of the anti-coproximinal subspaces in smooth Banach spaces. The geometry of such subspaces is further explored in finite-dimensional real polyhedral Banach spaces, revealing intriguing structural properties. Extending beyond finite dimensions, we analyze the problem in the space of scalar-valued continuous functions, where anti-coproximinal and strongly anti-coproximinal subspaces coincide, and offer a full characterization. We also examine the stability of these notions in the setting of vector-valued continuous functions. Finally, we explore these subspaces in the space of all bounded linear operators, further broadening the scope of our study.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Motivation and historical backgrounds

At the heart of mathematics lies the concept of distance, a seemingly simple measure that gives rise to rich and profound structures. From the early days of Euclidean geometry to the abstract landscapes of modern functional analysis, our understanding of space has evolved by refining how we compare and measure proximity between elements. The intuitive idea of "closeness" underpins both the philosophical and mathematical motivations for approximation theory. In the setting of Euclidean spaces, the problem of finding the *best approximation*, a point in a subspace closest to a given point outside the subspace, is directly tied to the notion of orthogonality. The classical picture is that the shortest path from a point to a subspace lies along the perpendicular, grounded in the basic geometric fact that, in a right-angled triangle, the perpendicular is shorter than the hypotenuse. Now interestingly, in a right angle triangle the base is also smaller than the hypotenuse, elevating this geometrical observation leads to the concept of *best coapproximation*, a dual notion to the best approximation.

In the general setting of Banach space the classical inner-product orthogonality breaks down. To address this limitation, generalized concepts such as *Birkhoff-James orthogonality* arise, capturing a geometric sense of orthogonality based on normed structure. While the theory of best approximation in Banach spaces has been extensively studied, the concept of best coapproximation remains less explored, yet it is an equally natural and meaningful counterpart. In this dissertation, we examine the best coapproximation problem through the lens of Birkhoff–James orthogonality, approaching it from both computational and analytical perspectives. We particularly focus on the existence of best coapproximations and investigate the newly introduced notions under the least favorable conditions for its existence.

1.2 Basic geometric properties

Let $\mathbb{X}$ be a Banach space over real or complex field $\mathbb{K}$ and let $\mathbb{X}^*$ be the dual of $\mathbb{X}$. We denote the notations $\mathbb{B}_{\mathbb{X}}$ and $S_{\mathbb{X}}$ for the unit ball and the unit sphere of $\mathbb{X}$, respectively. For a non-zero $x \in \mathbb{X}$, $x^* \in S_{\mathbb{X}^*}$ is said to be a supporting functional at x if $x^*(x) = \|x\|$. The set of all supporting functionals at x is denoted by $J(x)$, i.e., $J(x) = \{x^* \in S_{\mathbb{X}^*} : x^*(x) = \|x\|\}$. Let $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ ($\mathbb{K}(\mathbb{X}, \mathbb{Y})$) be the space of all bounded (compact) linear operators between $\mathbb{X}$ and $\mathbb{Y}$. Given $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, let M_T denote the norm attainment set of T, i.e., $M_T = \{x \in \mathbb{X} : \ \|x\| = 1, \|Tx\| = \|T\|\}$.

We now recall several geometric notions fundamental to the study of Banach spaces:

Definition 1.1. *Let $\mathbb{X}$ be a Banach space and let $x \in S_{\mathbb{X}}$.*

- ***Smooth point:*** *x is said to be a smooth point if $J(x)$ is singleton.*
- ***Extreme point:*** *x is said to be an extreme point of $B_{\mathbb{X}}$ if $x = (1-t)y + tz$, for some $t \in (0,1)$ and some $y, z \in B_{\mathbb{X}}$ implies that $x = y = z$. The set of all extreme points of $B_{\mathbb{X}}$ is denoted by $Ext(B_{\mathbb{X}})$.*
- ***Exposed point:*** *x is said to be an exposed point of $B_{\mathbb{X}}$ if there exists $x^* \in J(x)$ such that $x^*(y) < 1 = x^*(x)$, for any $y \in S_{\mathbb{X}} \setminus \{x\}$. Clearly, every exposed point of $B_{\mathbb{X}}$ is also an extreme point of $B_{\mathbb{X}}$. The set of all exposed points of $B_{\mathbb{X}}$ is denoted by $Exp(B_{\mathbb{X}})$.*
- ***Strongly exposed point:*** *We say x to be a strongly exposed point of $B_{\mathbb{X}}$ if there exists $x^* \in J(x)$ such that for any sequence $\{x_n\} \subset B_{\mathbb{X}}$, $x^*(x_n) \longrightarrow 1 = x^*(x)$ implies that $x_n \longrightarrow x$. Clearly, every strongly exposed point is an exposed point. The set of all strongly exposed points of $B_{\mathbb{X}}$ is denoted by st-$Exp(B_{\mathbb{X}})$.*
- ***Rotund point:*** *x is said to be a rotund point of $B_{\mathbb{X}}$ if for some $y \in B_{\mathbb{X}}$, $\|\frac{x+y}{2}\| = 2$ implies $x = y$. A smooth exposed point of $B_{\mathbb{X}}$ is a rotund point.*
- ***Face:*** *A convex set $F \subset S_{\mathbb{X}}$ is said to be a face of $B_{\mathbb{X}}$ if for any $y, z \in B_{\mathbb{X}}$, $\frac{1}{2}(y+z) \in F$ implies that $y, z \in F$. F is called a maximal face if for any face F' of $B_{\mathbb{X}}$, $F \subset F'$ implies $F = F'$. For $x \in \mathbb{X}$, $J(x)$ is a face of $B_{\mathbb{X}^*}$.*

These local geometric properties lead naturally to global properties of Banach spaces:

Definition 1.2. *Let $\mathbb{X}$ be a Banach space.*

- ***Smooth space:*** *The space $\mathbb{X}$ is said to be smooth if x is smooth for each $x \in S_{\mathbb{X}}$.*

- ***Strictly Banach space:*** *The space $\mathbb{X}$ is said to be strictly convex if $Ext(B_{\mathbb{X}}) = S_{\mathbb{X}}$, i.e., every element of $S_{\mathbb{X}}$ is an extreme point of $B_{\mathbb{X}}$. Note that in a strictly convex Banach space every element of its unit sphere is rotund.*

- ***Polyhedral Banach space:*** *A real finite-dimensional Banach space $\mathbb{X}$ is said to be polyhedral if $B_{\mathbb{X}}$ is a polyhedron. In other words, $\mathbb{X}$ is polyhedral if and only if $Ext(B_{\mathbb{X}})$ is finite. In a polyhedral Banach space every extreme point is exposed.*

We now recall two fundamental theorems in functional analysis that connect the geometry of Banach spaces with topological properties of dual spaces:

Theorem 1.1. ***(Banach-Alaoglu theorem:)*** *Let $\mathbb{X}$ be a Banach space. The unit ball $B_{\mathbb{X}^*}$ is compact in the weak*-topology.*

Theorem 1.2. ***(Krein-Milman theorem:)*** *Let A be a non-empty compact subset of a Hausdorff locally convex space E. Then $A = \overline{co(Ext(A))}$.*

Combining these two famous theorems we state the important property of the unit ball of dual space: The unit ball of a dual space $\mathbb{X}$ is the closed convex hull of its extreme points in weak*-topology, i.e., $B_{\mathbb{X}^*} = \overline{co(Ext(B_{\mathbb{X}^*}))}^{w*}$.

1.3 Best coapproximation

Before delving into the concept of best coapproximation, we first recall the classical notion of best approximation, a more extensively studied concept in approximation theory.

Definition 1.3. ***(Best approximation:)****[44] Let $\mathbb{X}$ be a Banach space and let $\mathbb{Y}$ be a subspace of $\mathbb{X}$. Given any $x \in \mathbb{X}$, we say that $y_0 \in \mathbb{Y}$ is a best approximation to x out of $\mathbb{Y}$ if $\|x - y_0\| \leq \|x - y\|$ for all $y \in \mathbb{Y}$.*

The set of all best approximation points to $x \in \mathbb{X}$ out of $\mathbb{Y}$ is denoted by $\mathcal{P}_{\mathbb{Y}}(x)$ (see [44]). In general, the existence and uniqueness of best approximations are not guaranteed in arbitrary Banach spaces. However, existence is ensured when $\mathbb{Y}$ is finite-dimensional, and uniqueness is guaranteed if $\mathbb{X}$ is a strictly convex Banach space. We now introduce the notion of best coapproximation, a dual-type concept to best approximation, with a fundamentally different geometric structure.

Definition 1.4. ***(Best coapproximation:)****[15, 33] Let $\mathbb{X}$ be a Banach space and let $\mathbb{Y}$ be a subspace of $\mathbb{X}$. Given any $x \in \mathbb{X}$, we say that $y_0 \in \mathbb{Y}$ is a best coapproximation to x out of $\mathbb{Y}$ if $\|y - y_0\| \leq \|x - y\|$ for all $y \in \mathbb{Y}$.*

Similar to the notion of best approximation, the existence and uniqueness of best coapproximations are not guaranteed in general. However, a key distinction between these two concepts becomes evident when we observe that, unlike best approximations, best coapproximations may fail to exist even in finite-dimensional settings. Given $x \in \mathbb{X}$ and a subspace $\mathbb{Y}$ of $\mathbb{X}$, the (possibly empty) set of all best coapproximations to x out of $\mathbb{Y}$ is denoted by $\mathcal{R}_{\mathbb{Y}}(x)$. The set *dom* $\mathcal{R}_{\mathbb{Y}}$ denotes the set of all points of $x \in \mathbb{X}$ from which the best coapproximation to x out of $\mathbb{Y}$ exists, i.e., *dom* $\mathcal{R}_{\mathbb{Y}} = \{x \in \mathbb{X} : \mathcal{R}_{\mathbb{Y}}(x) \neq \emptyset\}$. Before proceeding further, we note some basic characteristics of best coapproximation using the notion $\mathcal{R}_{\mathbb{Y}}$.

Theorem 1.3. *Let $\mathbb{Y}$ be a subspace of $\mathbb{X}$. Then the following holds true:*

(i) [15, Lemma 1] $\|\mathcal{R}_{\mathbb{Y}}(x)\| \leq \|x\|$, for any $x \in$ dom $\mathcal{R}_{\mathbb{Y}}$.

(ii) [33, Th. 5.3] If $x \in$ dom $\mathcal{R}_{\mathbb{Y}}$, then $\mathcal{R}_{\mathbb{Y}}(x) \subset$ dom $\mathcal{R}_{\mathbb{Y}}$ and $\mathcal{R}_{\mathbb{Y}}(\mathcal{R}_{\mathbb{Y}}(x)) = \mathcal{R}_{\mathbb{Y}}(x)$.

(iii) [33, Th. 5.3] If $x \in$ dom $\mathcal{R}_{\mathbb{Y}}$ and $y \in \mathbb{Y}$, then $x+y \in$ dom $\mathcal{R}_{\mathbb{Y}}$ and $\mathcal{R}_{\mathbb{Y}}(x+y) = \mathcal{R}_{\mathbb{Y}}(x)+y$.

(iv) [15, Lemma 1] If $x \in$ dom $\mathcal{R}_{\mathbb{Y}}$ and $\alpha \in \mathbb{C}$, then $\alpha x \in$ dom $\mathcal{R}_{\mathbb{Y}}$ and $\mathcal{R}_{\mathbb{Y}}(\alpha x) = \alpha \mathcal{R}_{\mathbb{Y}}(x)$.

(v) [15, Lemma 1] If $\mathbb{X}$ is smooth, then

(a) $\mathcal{R}_{\mathbb{Y}}(x)$ is singleton for each $x \in$ dom $\mathcal{R}_{\mathbb{Y}}$.

(b) dom $\mathcal{R}_{\mathbb{Y}}$ is a subspace of $\mathbb{X}$.

(vi) [15, Th. 1] If dim$(\mathbb{X}) \geq 3$, then $\mathbb{X}$ is a Hilbert space if and only if $\mathcal{R}_{\mathbb{Y}}(x) \neq \emptyset$, for each closed hyperspace $\mathbb{Y}$ of $\mathbb{X}$ and for each $x \in \mathbb{X}$.

Theorem 1.4. *[33, Th. 5.1, 5.2] Let $\mathbb{Y}, \mathbb{Z}$ be two subspaces of $\mathbb{X}$ with $\mathbb{Y} \subset \mathbb{Z}$. Then*

(i) $\mathcal{R}_{\mathbb{Y}}(\mathcal{R}_{\mathbb{Z}}(x)) \subset \mathcal{R}_{\mathbb{Y}}(x)$, for any $x \in \mathbb{X}$.

(ii) Whenever $\mathbb{X}$ is smooth, then for any $x \in$ dom $\mathcal{R}_{\mathbb{Z}}$, $\mathcal{R}_{\mathbb{Y}}(\mathcal{R}_{\mathbb{Z}}(x)) = \mathcal{R}_{\mathbb{Y}}(x)$. Moreover $\|\mathcal{R}_{\mathbb{Y}}(x)\| \leq \|\mathcal{R}_{\mathbb{Z}}(x)\|$, for any $x \in$ dom $\mathcal{R}_{\mathbb{Y}} \cap$ dom $\mathcal{R}_{\mathbb{Z}}$.

Since the existence and uniqueness of best coapproximations are not guaranteed in general, it is natural to focus on those subspaces for which these properties do hold. This consideration leads to the notions of coproximinal and co-Chebyshev subspaces.

Definition 1.5. *Let $\mathbb{Y}$ be a subspace of a Banach space $\mathbb{X}$.*

- ***Coproximinal subspace:*** *$\mathbb{Y}$ is said to be coproximinal if a best coapproximation to any element of $\mathbb{X}$ out of $\mathbb{Y}$ exists i.e., dom $\mathcal{R}_{\mathbb{Y}} = \mathbb{X}$.*

- ***Co-Chebyshev subspace:*** *A coproximinal subspace $\mathbb{Y}$ is said to be co-Chebyshev if the best coapproximation is unique for each point i.e., dom $\mathcal{R}_{\mathbb{Y}} = \mathbb{X}$ and $\mathcal{R}_{\mathbb{Y}}(x)$ is singleton, for each $x \in \mathbb{X}$.*

The existence of best coapproximation is deeply connected to the existence of a linear norm-1 projection, the connection is stated as follows.

Theorem 1.5. *[33] Let $\mathbb{Y}$ be a closed subspace of $\mathbb{X}$ and let $x \in \mathbb{X}$. Then the following statements are equivalent:*

(i) y_0 is the best coapproximation to x out of $\mathbb{Y}$, i.e., $y_0 \in \mathcal{R}_{\mathbb{Y}}(x)$.

(ii) There exists a linear norm-1 projection P from span$\{\mathbb{Y}, x\}$ to $\mathbb{Y}$ and $P(x) = \{y_0\}$.

This connection have been more extensively explored in [4, 5, 24, 26, 39, 43, 56]. The above connection indicates that a subspace $\mathbb{Y}$ is coproximinal in $\mathbb{X}$ if and only if there exists a linear norm-1 projection from every subspace of $\mathbb{X}$ containing $\mathbb{Y}$ to $\mathbb{Y}$. In this context it is important to note that a subspace $\mathbb{Y}$ of $\mathbb{X}$ is said to be 1-complemented if there exists a linear norm-1 projection map from $\mathbb{X}$ to $\mathbb{Y}$. Observing the connection between the existence of best coapproximation and linear norm-1 projection map, it is immediate that an 1-complemented subspace is a coproximinal subspace. However the converse is not true, which has been depicted by an example of Linedenstrauss, see [25].

1.4 Birkhoff-James orthogonality

Having introduced the concept of best coapproximation and explored some of its fundamental properties, we now turn our attention to a geometric notion that plays a pivotal role in our approach: *Birkhoff-James orthogonality*. This notion of orthogonality, which generalizes the classical idea from inner product spaces to arbitrary normed spaces, serves as a crucial analytical tool in the study of best coapproximation. We begin by recalling the definition which was introduced by Birkhoff [2].

Definition 1.6. ***(Birkhoff-James orthogonality:)****[2] Let $\mathbb{X}$ be a Banach space and let $x, y \in \mathbb{X}$. Then x is said to be Birkhoff-James orthogonal to y (denoted as $x \perp_B y$) if $\|x + \lambda y\| \geq \|x\|$, for all scalar λ.*

Building on Birkhoff's work, James [20, 19] provided this orthogonality notion in terms of supporting functionals, which elegantly illustrates the utility of this said orthogonality in studying the structural and analytical aspects of Banach spaces. The following theorem presents James's characterization in terms of elements of the dual space:

Theorem 1.6. *[20, Th. 2.1] Let $\mathbb{X}$ be a Banach space and let $x, y \in \mathbb{X}$. Then $x \perp_B y$ if and only if there exists $f \in J(x)$ such that $f(y) = 0$.*

Birkhoff-James orthogonality is homogeneous, i.e., $x \perp_B y \implies \alpha x \perp_B \beta y \; \forall \alpha, \beta \in \mathbb{C}$. However, in general, this notion of orthogonality is *not symmetric*, i.e., $x \perp_B y$ does not necessarily imply $y \perp_B x$. Notably, the symmetricity of Birkhoff-James orthogonality induces an inner product space on the space if the dimension of the space is greater or equal to 3. An important geometric consequence of James's result is that for any $x \in \mathbb{X}$, there exists a closed hyperspace $H \subset \mathbb{X}$ such that $x \perp_B H$, i.e., $x \perp_B h$ for all $h \in H$. However, the converse does not hold in general: given a closed hyperspace $H \subset \mathbb{X}$, there may not exist a nonzero element $x \in \mathbb{X}$ such that $x \perp_B H$. In fact, the existence of such an element for all hyperspaces characterizes *reflexivity* (see [19]). A Banach space $\mathbb{X}$ is reflexive if and only if for every closed hyperspace $H \subset \mathbb{X}$, there exists $x(\neq 0) \in \mathbb{X}$ such that $x \perp_B H$. In a similar vein, the dual condition characterizes *Hilbert spaces* (see [19]): A Banach space $\mathbb{X}$ is a Hilbert space if and only if for every closed hyperspace $H \subset \mathbb{X}$, there exists $x \neq 0 \in \mathbb{X}$ such that $H \perp_B x$. In this context, it is worth noting a striking example that illustrates the geometric irregularity of general Banach spaces (see [19]): In the space $C[0,1]$ of all continuous real-valued functions on $[0,1]$, there exists no closed hyperspace H and no nonzero $x \in C[0,1]$ such that $H \perp_B x$.

In the context of operator theory, Bhatia and Šemrl [8, Th. 1.1] and Paul [32, Lemma 2] separately provided a characterization of Birkhoff-James orthogonality in the space of bounded linear operators $\mathbb{L}(\mathbb{H})$, where $\mathbb{H}$ is a finite-dimensional Hilbert space.

Theorem 1.7. ***(Bhatia-Šemrl Theorem:)*** *Let $\mathbb{H}$ be a finite-dimensional Hilbert space and let $T, A \in \mathbb{L}(\mathbb{H})$. Then $T \perp_B A$ if and only if there exists $x \in M_T$ such that $\langle Tx, Ax \rangle = 0$.*

This result was later substantially generalized setting of Banach spaces by Sain and Paul [47, Th. 2.1], as stated below.

Theorem 1.8. *Let $\mathbb{X}$ be a finite-dimensional real Banach space. Let $T \in \mathbb{L}(\mathbb{X})$ be such that $M_T = \pm D$, where D is a closed, connected subset of $S_{\mathbb{X}}$. Then for any $A \in \mathbb{L}(\mathbb{X})$ with $T \perp_B A$, there exists $x \in D$ such that $Tx \perp_B Ax$.*

In addition to exact orthogonality, our study also requires the notion of approximate Birkhoff-James orthogonality, introduced by Chmieliński [9].

Definition 1.7. ***(Approximate Birkhoff-James orthogonality:)****[9, 10] Let $\mathbb{X}$ be a Banach space and let $x, y \in \mathbb{X}$. Let $\epsilon \in [0,1)$. Then x is said to be approximate Birkhoff-James orthogonal to y (denoted as $x \perp_B^{\epsilon} y$) if $\|x+\lambda y\| \geq \|x\| - \epsilon|\lambda|\|y\|$, for all scalar λ.*

The following characterization of this approximate version of this orthogonality in terms of functionals is particularly useful in our study.

Theorem 1.9. *[12, Th. 2.3] Let $\mathbb{X}$ be a Banach space. Suppose $\epsilon \in [0,1)$ and let $x, y \in \mathbb{X}$. Then $x \perp_B^{\epsilon} y$ if and only if there exists $f \in J(x)$ such that $|f(y)| \leq \epsilon\|y\|$.*

1.5 Birkhoff-James Orthogonality in the Study of Best Coapproximation

Let us first observe the connection between best approximation and Birkhoff-James orthogonality.

Theorem 1.10. *[44] Let $\mathbb{Y}$ be a subspace of $\mathbb{X}$ and let $x \in \mathbb{X}$. Then the following are equivalent:*

(i) y_0 is a best approximation to x out of $\mathbb{Y}$, i.e., $y_0 \in \mathcal{P}_{\mathbb{Y}}(x)$.

(ii) $x - y_0 \perp_B \mathbb{Y}$.

A similar connection exists between best coapproximation and Birkhoff-James orthogonality, as described in the following result:

Theorem 1.11. *[15] Let $\mathbb{Y}$ be a subspace of $\mathbb{X}$ and let $x \in \mathbb{X}$. Then the following are equivalent:*

(i) y_0 is a best coapproximation to x out of $\mathbb{Y}$, i.e., $y_0 \in \mathcal{R}_{\mathbb{Y}}(x)$.

(ii) $\mathbb{Y} \perp_B x - y_0$.

The orthogonality connection between these two notions best approximation and best coapproximation depicted the dual nature of these two problems. Since in general the Birkhoff-James orthogonality is not symmetric, for $x \in \mathbb{X}$, $\mathcal{P}_{\mathbb{Y}}(x) \neq \mathcal{R}_{\mathbb{Y}}(x)$. However, in case of Hilbert space, the orthogonality is symmetric and therefore, the notion of best coapproximation coincides with the notion of best approximation. Connecting Theorem 1.6 and Theorem 1.11, the following is immediate.

Theorem 1.12. *[33, Th. 2.1] Let $\mathbb{Y}$ be a subspace of $\mathbb{X}$ and let $x \in \mathbb{X} \setminus \mathbb{Y}$. Then y_0 is a best coapproximation to x out of $\mathbb{Y}$ if and only if for each $y \in \mathbb{Y}$, there exists an $f_y \in J(y)$ such that $f_y(x - y_0) = 0$.*

Building on this orthogonality framework, this dissertation explores the concept of best coapproximation within the setting of Banach spaces. The problem is examined from both computational and analytical perspectives across a range of geometrically distinct spaces, including smooth and strictly convex Banach spaces, polyhedral spaces (notably ℓ_1^n and ℓ_∞^n), spaces of continuous functions such as $C(K)$, and spaces of bounded linear operators. In the next section we give an outline of the content of thesis.

1.6 Outline of the thesis

This dissertation is organized into six chapters. The first chapter serves as an introduction, presenting fundamental notations and preliminary results concerning various geometric aspects of Banach spaces.

Chapter 2 explores the best coapproximation problem computationally to a given matrix T out of a given subspace $\mathbb{Y}$ of the space of all diagonal matrices $\mathcal{D}_n$. This is accomplished through the application of Birkhoff-James orthogonality techniques, supplemented by a newly introduced property termed as the $*$-Property. Notably, our approach yields a complete characterization of the best coapproximation problem in ℓ_∞^n as a specific case of our framework.

In Chapter 3, we study the best coapproximation problem in the space ℓ_1^n from a computational perspective. We present an algorithmic approach to solve the problem by studing a norming property of subspaces of ℓ_1^n, which helps us to provide a complete characterization of coproximinal and co-Chebyshev subspaces.

In Chapter 4, we introduce the concepts of anti-coproximinal and strongly anti-coproximinal subspaces to examine the least favorable scenarios encountered in the best coapproximation problem. Our primary focus is on finite-dimensional real polyhedral Banach spaces, specifically ℓ_1^n and ℓ_∞^n, aiming to explore the extreme properties of these newly introduced subspaces.

Chapter 5 delves into the study of anti-coproximinal and strongly anti-coproximinal subspaces within the space of continuous functions and vector valued continuous functions. A complete characterization has been provided for those subspaces in the space $C(K)$ and in the sequence spaces c_0, c, ℓ_∞. We also explore some necessary and sufficient conditions for strongly anti-coproximinal subspaces in general Banach space settings, separately.

The final chapter, i.e., Chapter 6 extends these two notions of subspaces to the space of all bounded linear operators. We also provide an abstract approach to tackle the best coapproximation problem in a more general setting, which helps us to obtain a complete characterization of coproximinal and co-Chebyshev subspaces as well as anti-coproximinal and strongly anti-coproximinal subspaces.

To make each chapter self-contained, we provide a brief motivation, along with the relevant notations and terminologies in the beginning of each chapter for the convenience of the readers.

CHAPTER 2

ON BEST COAPPROXIMATIONS IN SUBSPACES OF DIAGONAL MATRICES

2.1 Introduction

This chapter addresses the problem of best coapproximation in subspaces of the space of all diagonal matrices. Our aim is to develop a computational approach to solve the problem within these specific settings. The motivation for this investigation arises from recent progress in the study of approximation theory in Banach spaces, particularly employing orthogonality concept. Although the spirit of this work aligns with recent developments such as those in [45], our approach deviates significantly in its methods and focus, especially in the context of matrix subspaces. We begin by setting up the necessary notations and terminologies that will be used throughout the chapter.

We use the symbol $\mathbb{H}$ to denote a Hilbert space, along with its usual inner product $\langle \, , \rangle$ and its usual norm $||.||_2$. In this chapter, we will only work with *real* Hilbert spaces. Let $\perp$ denote

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the usual orthogonality relation on $\mathbb{H}$. $\mathbb{L}(\mathbb{H})$ ($\mathbb{K}(\mathbb{H})$) denotes the Banach space of all bounded (compact) linear operators on $\mathbb{H}$, endowed with the usual operator norm. Given $T \in \mathbb{L}(\mathbb{H})$, let M_T denote the norm attainment set of T,i.e, $M_T = \{x \in H \ : \ \|x\|_2 = 1, \|Tx\|_2 = \|T\|\}$. We note that $M_T \neq \emptyset$ whenever $T \in \mathbb{K}(\mathbb{H})$. In case $\mathbb{H}$ is finite-dimensional, given any $T \in \mathbb{L}(\mathbb{H})$, we identify T with its matrix representation with respect to the canonical basis of $\mathbb{H}$. Let $\mathcal{M}_n$ denote the space of all $n \times n$ real matrices and let $\mathcal{D}_n$ be the subspace of $\mathcal{M}_n$, consisting of diagonal matrices. Given $T \in \mathcal{M}_n$, let T^t denotes the transpose of T. Given any $A \in \mathcal{D}_n$ with diagonal entries $a_{ii}, \ 1 \leq i \leq n$, we write $A = ((a_{11}, a_{22}, \ldots, a_{nn}))$, for the sake of brevity. The zero element of $\mathbb{R}^n$ is denoted by θ, whenever $n > 1$.

Given $x \in \mathbb{X}$ and a subspace $\mathbb{Y}$ of $\mathbb{X}$, the (possibly empty) set of all best coapproximations to x out of $\mathbb{Y}$ is denoted by $\mathcal{R}_{\mathbb{Y}}(x)$. Our aim in this chapter is to explore the problem of finding the best coapproximation(s) to any given $T \in \mathcal{M}_n$ out of any given subspace $\mathbb{Y}$ of $\mathcal{D}_n$, provided the best coapproximation(s) exist. We employ Birkhoff-James orthogonality techniques and the concept of numerical range of an operator $T \in \mathbb{L}(\mathbb{H})$, to obtain a complete solution to the above problem, which is also computationally effective. Let us recall from the pioneering articles [2, 20] that given any two elements x, y in a Banach space $\mathbb{X}$, we say that x is Birkhoff-James orthogonal to y, written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$. It should be noted that given any subspace $\mathbb{Y}$ of a Banach space $\mathbb{X}$ and an element $x \in \mathbb{X}$, $y_0 \in \mathbb{Y}$ is a best coapproximation to x out of $\mathbb{Y}$ if and only if $\mathbb{Y} \perp_B (x - y_0,)$ i.e., $y \perp_B (x - y_0)$ for all $y \in \mathbb{Y}$. Using Theorem 1.1 of [8], also known as the Bhatia-Šemrl Theorem, we study the best coapproximation problem from the perspective of Birkhoff-James orthogonality. We also recall that given any $T \in \mathbb{L}(\mathbb{H})$, the numerical range of T is defined as $W(T) := \{\langle Tx, x\rangle : \|x\|_2 = 1\}$. We refer the readers to [17], for a comprehensive study and possible applications of the numerical range of an operator in $\mathbb{L}(\mathbb{H})$.

In this chapter, we obtain a complete characterization of the best coapproximation to an element of $\mathcal{M}_n$ out of a given subspace of $\mathcal{D}_n$. We emphasize that our method is computationally convenient and it is possible to present a tractable algorithmic solution to the above problem by using it. We further illustrate this by presenting explicit numerical examples in support of our claim. The first step in this direction is to obtain a theoretical characterization of the best coapproximation problem in $\mathbb{K}(\mathbb{H})$. The second step is to explore some fundamental attributes of the newly introduced $*$-Property in connection with the best coapproximation problem. In the final step, we assimilate the previously obtained results to present the desired algorithm to study the best coapproximation problem in any given subspace of $\mathcal{D}_n$. We also characterize the coproximinal subspaces and co-Chebyshev subspaces of $\mathcal{D}_n$ in $\mathcal{M}_n$. As another important application of the present study, we observe that a particular case of our method gives a complete solution to the best coapproximation problem in ℓ_∞^m, for any given $m \in \mathbb{N}$.

2.2 $*$-Property

2.2.1 Definition

In order to apply the above concepts in our designated study, we need to introduce the following definitions whose importance will be self-evident in due course of time.

Definition 2.1. *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be a set of linearly independent elements in $\mathcal{D}_n$, where $A_k = ((a_{11}^k, a_{22}^k, \ldots, a_{nn}^k))$, for each $1 \leq k \leq m$. Considering the diagonal matrices $A_1, A_2, \ldots, A_m$ as column vectors, we form the $n \times m$ matrix $\widetilde{A} = (\widetilde{a_{ij}})_{1 \leq i \leq n, 1 \leq j \leq m}$, where $\widetilde{a_{ij}} = a_{ii}^j$.*

(i) For each $i \in \{1, 2, \ldots, n\}$, the i-th component of $\mathcal{A}$ is defined as the i-th row of $\widetilde{A}$, i.e., $\left(a_{ii}^1, a_{ii}^2, \ldots, a_{ii}^m\right)$. Whenever the context is clear we simply say the i-th component of $\mathcal{A}$ as the i-th component.

(ii) The i-th component and the j-th component are said to be equivalent if

$$\left(a_{ii}^1, a_{ii}^2, \ldots, a_{ii}^m\right) = \pm \left(a_{jj}^1, a_{jj}^2, \ldots, a_{jj}^m\right).$$

(iii) The positively associated set $P_i^+(\mathcal{A})$ of the i-th component is defined as

$$P_i^+(\mathcal{A}) = \Big\{ j \in \{1, 2, \ldots, n\} : \left(a_{jj}^1, a_{jj}^2, \ldots, a_{jj}^m\right) = \left(a_{ii}^1, a_{ii}^2, \ldots, a_{ii}^m\right) \Big\}.$$

Similarly, the negatively associated set $P_i^-(\mathcal{A})$ is defined as

$$P_i^-(\mathcal{A}) = \Big\{ j \in \{1, 2, \ldots, n\} : \left(a_{jj}^1, a_{jj}^2, \ldots, a_{jj}^m\right) = -\left(a_{ii}^1, a_{ii}^2, \ldots, a_{ii}^m\right) \Big\}.$$

For simplicity, we write $P_i^+(\mathcal{A}) = P_i^+$ and $P_i^-(\mathcal{A}) = P_i^-$, if the context is clear.

(iv) The i-th component is said to satisfy the $*$-Property with respect to $\mathcal{A}$ if there exist $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$ such that

$$\Big| \sum_{k=1}^m \beta_k a_{ii}^k \Big| > \max \Big\{ \Big| \sum_{k=1}^m \beta_k a_{jj}^k \Big| : 1 \leq j \leq n,\ j \notin P_i^+ \cup P_i^- \Big\}.$$

Definition 2.2. *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be linearly independent in $\mathcal{D}_n$, where $A_k = ((a_{11}^k, a_{22}^k, \ldots, a_{nn}^k))$, for each $1 \leq k \leq m$. Suppose that the i-th component satisfies the $*$-Property with*

$|P_i^+ \cup P_i^-| = k_i$. *Given* $T = (b_{pq})_{1\le p,q\le n}$, *we define the* $*$*-associated matrix of* T *corresponding to the* i*-th component as a square matrix of order* k_i, *given by* ${}^iT_* = (c_{rs})_{1\le r,s\le k_i}$, *where*

$$\begin{aligned} c_{rs} &= b_{rs},\ (r,s) \in P_i^+ \times \left(P_i^+ \cup P_i^-\right) \\ &= -b_{rs}, (r,s) \in P_i^- \times \left(P_i^+ \cup P_i^-\right). \end{aligned}$$

2.2.2 Characteristics of the $*$-Property

We establish some fundamental attributes of the newly introduced $*$-Property which also plays an important role in our scheme. To begin with, we establish the basis invariance of equivalent components and the $*$-Property.

Proposition 2.1. *Let* Y *be a subspace of* $\mathcal{D}_n$ *and let* $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}, \mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ *be two bases of* Y, *where* $A_k = ((a_{11}^k, a_{22}^k, \ldots, a_{nn}^k))$ *and* $B_k = ((b_{11}^k, b_{22}^k, \ldots, b_{nn}^k))$, *for each* $1 \le k \le m$. *Then*
(i) $P_i^+(\mathcal{A}) = P_i^+(\mathcal{B})$ *and* $P_i^-(\mathcal{A}) = P_i^-(\mathcal{B}),\ i \in \{1, 2, \ldots, n\}$.
(ii) For any $i \in \{1, 2, \ldots, n\}$, *the* i*-th component satisfies the* $*$*-Property with respect to* $\mathcal{A}$ *if and only if the* i*-th component satisfies the* $*$*-Property with respect to* $\mathcal{B}$.

Proof. (i) Consider the two matrices $\widetilde{A}$ and $\widetilde{B}$ as constructed in Definition 2.1. Since $\mathcal{A}$ and $\mathcal{B}$ are two bases of Y, so there exists an invertible matrix $Q = (q_{ij})_{1\le i,j\le m}$ such that $\widetilde{A} = \widetilde{B}Q$, where $\widetilde{a_{ij}} = \sum_{k=1}^m \widetilde{b_{ik}} q_{kj}$, for any $1 \le i \le n, 1 \le j \le m$. The desired result then follows easily.
(ii) We first prove the necessary part. As before let $Q = (q_{ij})_{1\le i,j\le m}$ be the invertible matrix such that $\widetilde{A} = \widetilde{B}Q$, where $\widetilde{a_{ij}} = \sum_{k=1}^m \widetilde{b_{ik}} q_{kj}$, for any $1 \le i \le n, 1 \le j \le m$. Since the i-th component satisfies the $*$-Property with respect to $\mathcal{A}$, there exist $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$ such that

$$|\sum_{k=1}^m \beta_k a_{ii}^k| > \max\left\{|\sum_{k=1}^m \beta_k a_{jj}^k| : 1 \le j \le n,\ j \notin P_i^+ \cup P_i^-\right\}.$$

Observe that $\widetilde{A}\widetilde{\beta} = \widetilde{B}Q\widetilde{\beta}$, where $\widetilde{\beta} = (\beta_1\ \beta_2\ \ldots\ \beta_m)^t$. Considering $\widetilde{\gamma} = (\gamma_1\ \gamma_2\ \ldots\ \gamma_m)^t = Q\widetilde{\beta}$, it is easy to see that for any $r \in \{1, 2, \ldots, n\}$,

$$|\sum_{k=1}^m \gamma_k b_{rr}^k| = |\sum_{k=1}^m \left(\sum_{j=1}^m q_{kj}\beta_j\right)\widetilde{b_{rk}}| = |\sum_{j=1}^m (\sum_{k=1}^m \widetilde{b_{rk}} q_{kj})\beta_j| = |\sum_{j=1}^m \widetilde{a_{rj}}\beta_j| = |\sum_{j=1}^m \beta_j a_{rr}^j|.$$

This immediately shows that the i-th component satisfies the $*$-Property with respect to $\mathcal{B}$. This completes the necessary part. The sufficient part follows similarly. □

In light of the above theorem, from now onwards we will not explicitly mention the choice of basis in the description of the $*$-Property. Our next theorem essentially guarantees the existence of the $*$-Property.

Theorem 2.1. *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be linearly independent in $\mathcal{D}_n$, where $A_k = ((a_{11}^k, a_{22}^k, \ldots, a_{nn}^k))$, for each $1 \leq k \leq m$. Then there exists $1 \leq i \leq n$ such that the i-th component satisfies the $*$-Property.*

Proof. Let the i_1-th, i_2-th, $\ldots$, i_p-th components represent all the nonequivalent components For any $\widetilde{w} = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathbb{R}^m$, consider the set of scalars

$$S_{\widetilde{w}} := \left\{ |\sum_{k=1}^{m} \gamma_k a_{i_1 i_1}^k|, |\sum_{k=1}^{m} \gamma_k a_{i_2 i_2}^k|, \ldots, |\sum_{k=1}^{m} \gamma_k a_{i_p i_p}^k| \right\}.$$

Case 1 : If $S_{\widetilde{w}}$ attains its maximum at a unique point, say at $|\sum_{k=1}^{m} \gamma_k a_{i_r i_r}^k|$, where $r \in \{1, 2, \ldots, p\}$, then clearly the i_r-th component satisfies the $*$-Property.

Case 2 : Let us assume that the maximum of $S_{\widetilde{w}}$ is attained at exactly two points. Suppose that for $i_s, i_t \in \{i_1, i_2, \ldots, i_p\}$ and $i_s \neq i_t$,

$$|\sum_{k=1}^{m} \gamma_k a_{i_s i_s}^k| = |\sum_{k=1}^{m} \gamma_k a_{i_t i_t}^k| > \max \left\{ |\sum_{k=1}^{m} \gamma_k a_{qq}^k| : q \in \{i_1, i_2, \ldots, i_p\} \setminus \{i_s, i_t\} \right\}.$$

Let us define functions $f_s, f_t : \mathbb{R}^m \longrightarrow \mathbb{R}$ given by

$$f_s(\widetilde{u}) := |\langle\, \widetilde{u}, \widetilde{a}_{i_s} \,\rangle| \text{ and } f_t(\widetilde{u}) := |\langle\, \widetilde{u}, \widetilde{a}_{i_t} \,\rangle|,$$

where $\widetilde{u} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m$ and $\widetilde{a}_{i_s} = (a_{i_s i_s}^1, a_{i_s i_s}^2, \ldots, a_{i_s i_s}^m), \widetilde{a}_{i_t} = (a_{i_t i_t}^1, a_{i_t i_t}^2, \ldots, a_{i_t i_t}^m)$ are the i_s-th and the i_t-th component, respectively.

Let us also define another function, $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ given by

$$g(\widetilde{u}) := \max \{ |\langle\, \widetilde{u}, \widetilde{a}_q \,\rangle| : q \in \{i_1, i_2, \ldots, i_p\} \setminus \{i_s, i_t\} \},$$

where $\widetilde{a}_q = (a_{qq}^1, a_{qq}^2, \ldots, a_{qq}^m)$ is the q-th component. Since f_i, g are continuous function on $\mathbb{R}^m$, $\phi_i = f_i - g$ is also continuous and $\phi_i(\widetilde{w}) > 0$, for all $i \in \{s, t\}$, where $\widetilde{w} = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathbb{R}^m$. It is easy to observe that there exists an open ball $\mathcal{B}_\delta(\widetilde{w})$, with radius $\delta > 0$ and centered at $\widetilde{w}$, such that $\phi_i(\widetilde{y}) > 0$, for all $\widetilde{y} \in \mathcal{B}_\delta(\widetilde{w})$. Consider the hyperspaces H_1, H_2 of $\mathbb{R}^m$ given by

$$H_1 = \{ \widetilde{x} \in \mathbb{R}^m : \langle\, \widetilde{x}, (\widetilde{a}_{i_s} + \widetilde{a}_{i_t}) \rangle = 0 \},$$

$$H_2 = \{ \widetilde{x} \in \mathbb{R}^m : \langle\, \widetilde{x}, (\widetilde{a}_{i_s} - \widetilde{a}_{i_t}) \rangle = 0 \}.$$

We note that $\{\widetilde{x} \in \mathbb{R}^m : f_s(\widetilde{x}) = f_t(\widetilde{x})\} = H_1 \cup H_2$, which is a nowhere dense set in $\mathbb{R}^m$. Therefore, by choosing $\widetilde{v} := (\beta_1, \beta_2, \ldots, \beta_m) \in \mathcal{B}_\delta(\widetilde{w}) \setminus (H_1 \cup H_2)$, we obtain that $f_s(\widetilde{v}) \neq f_t(\widetilde{v})$. Without loss of generality, assume $f_s(\widetilde{v}) > f_t(\widetilde{v})$. It is now easy to observe that

$$|\sum_{k=1}^{m} \beta_k a^k_{i_s i_s}| > \max\left\{|\sum_{k=1}^{m} \beta_k a^k_{qq}| : 1 \leq q \leq n,\ q \notin P^+_{i_s} \cup P^-_{i_s}\right\}.$$

Therefore, the i_s-th component satisfies the $*$-Property.

Case 3 : Suppose that the maximum of $S_{\widetilde{w}}$ is attained at $r(> 2)$ number of points and let the i_1-th,i_2-th, $\ldots$, i_r-th components satisfy

$$|\sum_{k=1}^{m} \gamma_k a^k_{i_1 i_1}| = \ldots = |\sum_{k=1}^{m} \gamma_k a^k_{i_r i_r}| > \max\left\{|\sum_{k=1}^{m} \gamma_k a^k_{qq}| : q \in \{i_1, \ldots, i_p\} \setminus \{i_1, \ldots, i_r\}\right\}.$$

By similar argument as given in Case 2, it can be shown that at least one of the i_l-th components satisfies the $*$-Property, where $l \in \{1, 2, \ldots, r\}$. This completes the theorem. □

Remark 2.2. *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be linearly independent in $\mathcal{D}_n$, where $A_k = ((a^k_{11}, a^k_{22}, \ldots, a^k_{nn}))$, for each $1 \leq k \leq m$. In particular, for any $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$, there exists an i-th component such that $\|\sum_{k=1}^m \beta_k A_k\| = |\sum_{k=1}^m \beta_k a^k_{ii}|$, where the i-th component satisfies the $*$-Property.*

Our next aim is to obtain a tractable necessary and sufficient condition for the $*$-Property. In this context, we first recall the definition of a normal cone. A non-empty set $K \subset \mathbb{R}^n$ is said to be a normal cone if the following three conditions are satisfied:

$$(i) u, v \in K \Rightarrow u + v \in K, (ii) u \in K,\ \alpha \geq 0 \Rightarrow \alpha u \in K, (iii) K \cap (-K) = \{\theta\}.$$

We define interior of the normal cone K, denoted by $int(K)$, as the collection of all interior points of the normal cone K. We refer the readers to [50], for an application of the notion of normal cones in studying approximate Birkhoff-James orthogonality in Banach spaces. We also require the following lemma for our purpose.

Lemma 2.1. *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be linearly independent in $\mathcal{D}_n$, where $A_k = ((a^k_{11}, a^k_{22}, \ldots, a^k_{nn}))$, for each $1 \leq k \leq m$. Given any $i, j \in \{1, 2, \ldots, n\}$, where $j \notin P^+_i \cup P^-_i$, there exist a pair of normal cones whose interiors are the collection of all the $(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{R}^m$ such that $|\sum_{k=1}^m \beta_k a^k_{ii}| > |\sum_{k=1}^m \beta_k a^k_{jj}|$.*

Proof. Let us consider the set

$$C := \left\{ (\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{R}^m : \ |\sum_{k=1}^{m} \beta_k a_{ii}^k| > |\sum_{k=1}^{m} \beta_k a_{jj}^k| \right\}.$$

Let us also construct two sets K_1 and K_2 such that

$$K_1 := \left\{ (\beta_1, \beta_2, \ldots, \beta_m) \in C : \ \sum_{k=1}^{m} \beta_k a_{ii}^k > 0 \right\} \bigcup \{\theta\},$$

$$K_2 := \left\{ (\beta_1, \beta_2, \ldots, \beta_m) \in C : \ \sum_{k=1}^{m} \beta_k a_{ii}^k < 0 \right\} \bigcup \{\theta\}.$$

From the definition of K_1 and K_2, it is evident that $K_1 = -K_2$. Now, it is immediate that $\widetilde{x} \in K_1$ implies that $\alpha\widetilde{x} \in K_1$, for all $\alpha \geq 0$. Therefore, to prove that K_1 is a normal cone, we only need to show $\widetilde{u}, \widetilde{v} \in K_1$ implies $\widetilde{u} + \widetilde{v} \in K_1$. Suppose that $\widetilde{u} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $\widetilde{v} = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in K_1$. Then, $\sum_{k=1}^{m}(\alpha_k + \gamma_k)a_{ii}^k > 0$ and for any $j \in \{1, 2, \ldots, n\}$ such that $j \notin P_i^+ \cup P_i^-$, it follows that

$$\begin{aligned} |\sum_{k=1}^{m}(\alpha_k + \gamma_k)a_{jj}^k| \leq |\sum_{k=1}^{m} \alpha_k a_{jj}^k| + |\sum_{k=1}^{m} \gamma_k a_{jj}^k| \quad &< \quad |\sum_{k=1}^{m} \alpha_k a_{ii}^k| + |\sum_{k=1}^{m} \gamma_k a_{ii}^k| \\ &= \quad |\sum_{k=1}^{m}(\alpha_k + \gamma_k)a_{ii}^k|. \end{aligned}$$

This proves that K_1 (and therefore, K_2) is a normal cone. It is rather straightforward to verify that $int(K_1) = K_1 \setminus \{\theta\}$ and $int(K_2) = K_2 \setminus \{\theta\}$. Therefore, $int(K_1) \cup int(K_2) = C$, as desired. This completes the proof of the lemma. □

Next we introduce the notion of associated pair of cones, which turns out to be useful in characterizing the $*$-Property.

Definition 2.3. *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be linearly independent in $\mathcal{D}_n$, where $A_k = ((a_{11}^k, a_{22}^k, \ldots, a_{nn}^k))$, for each $1 \leq k \leq m$. Given any $i \in \{1, 2, \ldots, n\}$, we define the pair of normal cones $K_j^i, -K_j^i$ as the associated pair of cones of the i-th component with respect to the j-th component, given by*

$$K_j^i := \left\{ (\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{R}^m : \ |\sum_{k=1}^{m} \beta_k a_{ii}^k| > |\sum_{k=1}^{m} \beta_k a_{jj}^k| \text{ and } \sum_{k=1}^{m} \beta_k a_{ii}^k > 0 \right\} \cup \{\theta\},$$

for all $j \notin P_i^+ \cup P_i^-$.

Finally, we are in a position to characterize the $*$-Property from a geometric perspective.

Theorem 2.3. *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be linearly independent in $\mathcal{D}_n$, where $A_k = ((a^k_{11}, a^k_{22}, \ldots, a^k_{nn}))$, for each $1 \leq k \leq m$. Then for any $i \in \{1, 2, \ldots, n\}$, the i-th component satisfies the $*$-Property if and only if*

$$\bigcap \left\{ int(K^i_j) \cup int(-K^i_j) : 1 \leq j \leq n, j \notin P^+_i \cup P^-_i \right\} \neq \emptyset.$$

Proof. Suppose that the i-th component satisfies the $*$-Property, i.e., there exists $\widetilde{x} = (\beta_1, \beta_1, \ldots, \beta_m) \in \mathbb{R}^m$ such that

$$|\sum_{k=1}^{m} \beta_k a^k_{ii}| > max \left\{ |\sum_{k=1}^{m} \beta_k a^k_{jj}| : j \notin P^+_i \cup P^-_i \right\}.$$

This is equivalent to $\widetilde{x} \in int(K^i_j) \cup int(-K^i_j)$, for all $1 \leq j \leq n$ and $j \notin P^+_i \cup P^-_i$, where $K^i_j, -K^i_j$ are the pair of associated cones of the i-th component with respect to the j-th component. Therefore,

$$\bigcap \left\{ int(K^i_j) \cup int(-K^i_j) : 1 \leq j \leq n, j \notin P^+_i \cup P^-_i \right\} \neq \emptyset.$$

This completes the proof of the necessary part of the theorem. We note that the sufficient part of theorem also follows from similar arguments and the definition of pair of associated cones. This establishes the theorem.

□

We next obtain a simple and useful sufficient condition for the $*$-Property. It should be noted that in practise, the following result can be readily applied in most cases of the computations involving the $*$-Property, since checking the linear independence of a given set of vectors is not complicated at all by virtue of the well-known method of row reduction of matrices.

Proposition 2.2. *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be linearly independent in $\mathcal{D}_n$, where $A_k = ((a^k_{11}, a^k_{22}, \ldots, a^k_{nn}))$, for each $1 \leq k \leq m$. Suppose that the i-th component $\left(a^1_{ii}, a^2_{ii}, \ldots, a^m_{ii}\right) \notin span\{\left(a^1_{jj}, a^2_{jj}, \ldots, a^m_{jj}\right) : 1 \leq j \leq n,\ j \notin P^+_i \cup P^-_i\}$, where $\left(a^1_{jj}, a^2_{jj}, \ldots, a^m_{jj}\right)$ is the j-th component. Then the i-th component satisfies the $*$-Property.*

Proof. Let

$$\begin{aligned} Y_1 &= span\left\{ \left(a^1_{jj}, a^2_{jj}, \ldots, a^m_{jj}\right) : 1 \leq j \leq n,\ j \notin P^+_i \cup P^-_i \right\}, \\ Y_2 &= span\left\{ \left(a^1_{jj}, a^2_{jj}, \ldots, a^m_{jj}\right) : 1 \leq j \leq n \right\}. \end{aligned}$$

Clearly, $Y_1 \subsetneq Y_2 = \mathbb{R}^m$, which implies that $Y_2^{\perp} \subsetneq Y_1^{\perp}$. Therefore, there exists $(\gamma_1, \gamma_2, \ldots, \gamma_m) \in$

$Y_1^{\perp} \setminus Y_2^{\perp}$ such that

$$|\sum_{k=1}^{m} \gamma_k a_{ii}^k| > \max\left\{|\sum_{k=1}^{m} \gamma_k a_{jj}^k| : 1 \leq j \leq n,\ j \notin P_i^+ \cup P_i^-\right\} = 0.$$

In other words, the i-th component satisfies the $*$-Property, as desired. □

Remark 2.4. *Suppose that $\mathcal{T}_i$ is the collection of all those j such that the j-th component is a scalar multiple of the i-th component, where $i, j \in \{1, 2, \ldots, n\}$. Let us assume that the i-th component $(a_{ii}^1, a_{ii}^2, \ldots, a_{ii}^m) = c_j(a_{jj}^1, a_{jj}^2, \ldots, a_{jj}^m)$, where $\left(a_{jj}^1, a_{jj}^2, \ldots, a_{jj}^m\right)$ is the j-th component and $|c_j| \geq 1$, for all $j \in \mathcal{T}_i$. Also assume that $(a_{ii}^1, a_{ii}^2, \ldots, a_{ii}^m) \notin span\{\left(a_{kk}^1, a_{kk}^2, \ldots, a_{kk}^m\right) : 1 \leq k \leq n,\ \ k \notin \mathcal{T}_i\}$. Following similar argument from Proposition 2.2, the i-th component satisfies the $*$-Property.*

The following lemma is crucial for that purpose, besides being interesting in its own right by providing a lower bound on the number of nonequivalent components satisfying the $*$-Property.

Lemma 2.2. *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be linearly independent in $\mathcal{D}_n$, where $A_k = ((a_{11}^k, a_{22}^k, \ldots, a_{nn}^k))$, for each $1 \leq k \leq m$. Let the total number of nonequivalent components satisfying the $*$-Property be p. Then $p \geq m$.*

Proof. Suppose that the j_1-th, j_2-th, $\ldots$, j_p-th nonequivalent components satisfy the $*$-Property. Suppose on the contrary that $p < m$. Let $Y_1 = span\{(a_{j_sj_s}^1, a_{j_sj_s}^2, \ldots, a_{j_sj_s}^m) : 1 \leq s \leq p\}$ and let $Y_2 = span\{\left(a_{ii}^1, a_{ii}^2, \ldots, a_{ii}^m\right)) : 1 \leq i \leq n\}$. Clearly, $Y_1 \subsetneq Y_2 = \mathbb{R}^m$, which implies that $Y_2^{\perp} \subsetneq Y_1^{\perp}$. Therefore, there exists $(\gamma_1, \gamma_2, \ldots, \gamma_m) \in Y_1^{\perp} \setminus Y_2^{\perp}$ such that

$$|\sum_{k=1}^{m} \gamma_k a_{ii}^k| > \max\left\{|\sum_{k=1}^{m} \gamma_k a_{j_sj_s}^k| : 1 \leq s \leq p\right\} = 0,$$

for some $i \notin \{j_1, j_2, \ldots, j_p\}$. Following Theorem 2.1, we obtain that the i-th component, which is nonequivalent to the j_1-th, j_2-th, $\ldots$, j_p-th components, satisfies the $*$-Property. This contradiction completes the proof of the lemma.

□

2.3 Best coapproximation in subspaces of $\mathcal{D}_n$

We begin with a theoretical characterization of best coapproximations in $\mathbb{K}(\mathbb{H})$, that will play a crucial role in the computational approach towards finding best coapproximation(s) (provided it exists) in any given subspace of $\mathcal{D}_n$, as adopted in the present chapter.

Theorem 2.5. *Let $\mathbb{H}$ be a Hilbert space and let $T, A_1, A_2, \ldots, A_m \in \mathbb{K}(\mathbb{H})$. Given any $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$, $\sum_{i=1}^{m} \alpha_i A_i$ is a best coapproximation to T out of span$\{A_1, A_2, \ldots, A_m\}$ if and only if given any $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$, there exists $x \in M_{\sum_{i=1}^{m} \beta_i A_i}$ such that*

$$\langle \sum_{i=1}^{m} \beta_i A_i x, (T - \sum_{i=1}^{m} \alpha_i A_i) x \rangle = 0.$$

Proof. It follows from the definitions of Birkhoff-James orthogonality and best coapproximation that $\sum_{i=1}^{m} \alpha_i A_i$ is a best coapproximation to T out of *span*$\{A_1, A_2, \ldots, A_m\}$ if and only if $A \perp_B (T - \sum_{i=1}^{m} \alpha_i A_i)$, for all $A \in$ *span* $\{A_1, A_2, \ldots, A_m\}$. Clearly, this is equivalent to the following:

$$\sum_{i=1}^{m} \beta_i A_i \perp_B (T - \sum_{i=1}^{m} \alpha_i A_i) \ \forall \ \beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}.$$

It follows from [47, Th. 2.2] that for any $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$, $M_{\sum_{i=1}^{m} \beta_i A_i}$ is the unit sphere of some subspace of $\mathbb{H}$. Now applying [46, Th. 2.2], we conclude that the above condition is equivalent to the existence of $x = x(\beta_1, \ldots, \beta_m) \in M_{\sum_{i=1}^{m} \beta_i A_i}$ such that $\langle \sum_{i=1}^{m} \beta_i A_i x, (T - \sum_{i=1}^{m} \alpha_i A_i) x \rangle = 0$, for any $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$. This completes the proof of the theorem. □

We are now ready to present a computationally convenient characterization of the best coapproximation to an element of $\mathcal{M}_n$ out of a given subspace of $\mathcal{D}_n$.

Theorem 2.6. *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be linearly independent in $\mathcal{D}_n$, where $A_k = ((a_{11}^k, a_{22}^k, \ldots, a_{nn}^k))$, for each $1 \leq k \leq m$. Suppose that the j_1-th, j_2-th, $\ldots, j_r$-th nonequivalent components satisfy the $*$-Property. Then given any $T \in \mathcal{M}_n$, $\sum_{k=1}^{m} \alpha_k A_k$ is a best coapproximation to T out of span $\{A_1, A_2, \ldots, A_m\}$ if and only if $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ satisfy the following relations:*

$$a_{j_p j_p}^1 \alpha_1 + a_{j_p j_p}^2 \alpha_2 + \ldots + a_{j_p j_p}^m \alpha_m \in W\left({}^{j_p}T_*\right),$$

for all $p \in \{1, 2, \ldots, r\}$, where $W\left({}^{j_p}T_*\right)$ is the numerical range of the $*$-associated matrix of T corresponding to the j_p-th component.

Proof. Let us first prove the necessary part of the theorem. Assume that the j_s-th component satisfies the $*$-Property, where $s \in \{1, 2, \ldots, r\}$. Then there exists $\widetilde{v} = (\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{R}^m$ such that $|\sum_{k=1}^{m} \beta_k a_{j_s j_s}^k| > \max\{|\sum_{k=1}^{m} \beta_k a_{qq}^k| : 1 \leq q \leq n, \ q \notin P_{j_s}^+ \cup P_{j_s}^-\}$. Therefore, $\| \sum_{k=1}^{m} \beta_k A_k \| = |\sum_{k=1}^{m} \beta_k a_{j_s j_s}^k|$ and the norm attainment set of $\sum_{k=1}^{m} \beta_k A_k$ is

$$M_{\sum_{k=1}^{m} \beta_k A_k} = \left\{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \ : \ \|x\|_2 = 1, x_h = 0 \ \forall \ h \notin P_{j_s}^+ \cup P_{j_s}^- \right\}.$$

Let $T = (b_{uv})_{1\le u,v\le n} \in \mathcal{M}_n$. Since $\sum_{k=1}^{m} \alpha_k A_k$ is a best coapproximation to T out of $span\{A_1, A_2, \ldots, A_m\}$, it follows from Theorem 2.5 that for some $x \in M_{\sum_{k=1}^{m}\beta_k A_k}$,

$$\begin{aligned} &\left\langle \sum_{k=1}^{m} \beta_k A_k x,\ (T - \sum_{k=1}^{m} \alpha_k A_k)x \right\rangle = 0 \\ \implies &\left\langle \sum_{k=1}^{m} \beta_k A_k x,\ Tx \right\rangle = \left\langle \sum_{k=1}^{m} \beta_k A_k x,\ \sum_{k=1}^{m} \alpha_k A_k x \right\rangle. \end{aligned}$$

By a straight forward calculation, the previous equation can be expressed as

$$\begin{aligned} a^1_{j_s j_s}\alpha_1 + \ldots + a^m_{j_s j_s}\alpha_m &= \sum_{u\in P^+_{j_s}, v\in P^+_{j_s}\cup P^-_{j_s}} b_{uv}x_u x_v - \sum_{u\in P^-_{j_s}, v\in P^+_{j_s}\cup P^-_{j_s}} b_{uv}x_u x_v \\ &\in W\left({}^{j_s}T_*\right). \end{aligned}$$

Similarly, we can observe that for all $p \in \{1, 2, \ldots, r\}$,

$$a^1_{j_p j_p}\alpha_1 + a^2_{j_p j_p}\alpha_2 + \ldots + a^m_{j_p j_p}\alpha_m \in W\left({}^{j_p}T_*\right), \tag{2.1}$$

completing the proof of the necessary part.

We now prove the sufficient part of the theorem. For any $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$, not all zero, by virtue of Remark 2.2, there exists $t \in \{1, 2, \ldots, r\}$ such that the j_t-th component satisfies $\|\sum_{k=1}^{m}\beta_k A_k\| = |\sum_{k=1}^{m}\beta_k a^k_{j_t j_t}|$. Let $P^+_{j_t} = \{j_t, j_{t_2}, \ldots, j_{t_w}\}$ and $P^-_{j_t} = \{j_{t_{w+1}}, j_{t_{w+2}}, \ldots, j_{t_v}\}$, so that $|P^+_{j_t} \cup P^-_{j_t}| = v$. From the hypothesis, $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ satisfy the following relations:

$$a^1_{j_t j_t}\alpha_1 + a^2_{j_t j_t}\alpha_2 + \ldots + a^m_{j_t j_t}\alpha_m \in W\left({}^{j_t}T_*\right).$$

Therefore, there exists $y = (y_t, y_{t_2}, \ldots, y_{t_v}) \in \mathbb{R}^v$ with $\|y\|_2 = 1$ such that

$$a^1_{j_t j_t}\alpha_1 + a^2_{j_t j_t}\alpha_2 + \ldots + a^m_{j_t j_t}\alpha_m = \langle {}^{j_t}T_* y,\ y\rangle. \tag{2.2}$$

Now by taking $\widehat{y} = (\widetilde{y}_1, \widetilde{y}_2, \ldots, \widetilde{y}_n) \in \mathbb{R}^n$ such that $\widetilde{y}_h = 0\ \forall\ j_h \notin P^+_{j_t} \cup P^-_{j_t}$ and $\widetilde{y}_h = y_h\ \forall\ j_h \in P^+_{j_t} \cup P^-_{j_t}$, it is easy to observe that $\widehat{y} \in M_{\sum_{k=1}^{m}\beta_k A_k}$. By some easy calculations and by using the equation (2.2), we conclude that

$$\left\langle \sum_{k=1}^{m} \beta_k A_k \widehat{y},\ (T - \sum_{k=1}^{m} \alpha_k A_k)\ \widehat{y} \right\rangle = (\sum_{k=1}^{m} \beta_k a^k_{j_t j_t}) \left[\langle y,\ {}^{j_t}T_* y\rangle - \sum_{k=1}^{m} \alpha_k a^k_{j_t j_t} \right] = 0.$$

The sufficient part of the theorem now follows directly from Theorem 2.5. This establishes the theorem.

□

Remark 2.7. *Suppose that $A_1, A_2, \ldots, A_m \in \mathcal{M}_n$, where $1 \leq m \leq n$, are such that $A_iA_j^t, A_i^tA_j$ are symmetric, for all $i, j \in \{1, 2, \ldots, m\}$. Then from [28, Cor. 9], there exist orthogonal matrices P and Q such that $P^tA_iQ = D_i$, where $D_i \in \mathcal{D}_n$, for all $i \in \{1, 2, \ldots, m\}$. Moreover, since P and Q are orthogonal matrices, it is easy to see that $\| \sum_{i=1}^m \beta_i A_i \| = \| \sum_{i=1}^m \beta_i D_i \|$, for all $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$.*

The above remark allows us to present the following strengthened version of Theorem 2.6.

Theorem 2.8. *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be linearly independent in $\mathcal{M}_n$ such that $A_iA_j^t, A_i^tA_j$ are symmetric, for all $i, j \in \{1, 2, \ldots, m\}$. Let $D_1, D_2, \ldots, D_m \in \mathcal{D}_n$ be such that $D_i = P^tA_iQ$, where $D_i = ((d_{11}^i, d_{22}^i, \ldots, d_{nn}^i))$, for all $i \in \{1, 2, \ldots, m\}$ and $P, Q \in \mathcal{M}_n$ are orthogonal matrices. Suppose that the j_1-th, j_2-th, $\ldots$, j_r-th nonequivalent components satisfy the $*$-Property (with respect to span$\{D_1, D_2, \ldots, D_m\}$). Then given any $T \in \mathcal{M}_n$, $\sum_{i=1}^m \alpha_i A_i$ is a best coapproximation to T out of span$\{A_1, A_2, \ldots, A_m\}$ if and only if $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ satisfy the following relations:*

$$d_{j_pj_p}^1\alpha_1 + d_{j_pj_p}^2\alpha_2 + \ldots + d_{j_pj_p}^m\alpha_m \in W\left({}^{j_p}(P^tTQ)_*\right),$$

for all $p \in \{1, 2, \ldots, r\}$, where $W\left({}^{j_p}(P^tTQ)_*\right)$ is the numerical range of the $*$-associated matrix of P^tTQ corresponding to the j_p-th component.

Proof. $\sum_{i=1}^m \alpha_i A_i$ is a best coapproximation to T out of *span*$\{A_1, A_2, \ldots, A_m\}$ if and only if given any $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$, there exists $x \in M_{\sum_{i=1}^m \beta_i A_i}$ such that
$\langle\, (\sum_{i=1}^m \beta_i A_i)\, x, (T - \sum_{i=1}^m \alpha_i A_i)\, x \,\rangle = 0$, i.e,

$$\left\langle \left(P\sum_{i=1}^m \beta_i D_i Q^t\right) x, \left(T - P\sum_{i=1}^m \alpha_i D_i Q^t\right) x \right\rangle = 0.$$

So, for $y = Q^tx$, it is immediate that $\langle\, (\sum_{i=1}^m \beta_i D_i)\, y, \left(P^tTQ - \sum_{i=1}^m \alpha_i D_i\right) y \,\rangle = 0$. We also note that $x \in M_{\sum_{i=1}^m \beta_i A_i}$ if and only if $y = Q^tx \in M_{\sum_{i=1}^m \beta_i D_i}$. Therefore, $\sum_{i=1}^m \alpha_i A_i$ is a best coapproximation to T out of *span*$\{A_1, A_2, \ldots, A_m\}$ if and only if $\sum_{i=1}^m \alpha_i D_i$ is a best coapproximation to P^tTQ out of *span*$\{D_1, D_2, \ldots, D_m\}$. Now the desired result follows directly from Theorem 2.6. This completes the proof of the theorem. □

To illustrate the applicability of Theorem 2.6 from a computational point of view, we next present a series of explicit numerical examples elaborating the different features of the best coapproximation problem, related to the existence and the uniqueness. In each case, an algorithmic approach is presented which further underlines the usefulness of the $*$-Property in studying best coapproximation problems in subspaces of $\mathcal{D}_n$.

Example 2.9. *Let* $A_1 = ((\ 7,\ -5,\ 2,\ 6,\ -7,\ -5,\ 1\ ))$, $A_2 = ((\ 1,\ 3,\ 4,\ 3,\ -1,\ 3,\ 2\ ))$, $A_3 = ((\ 3,\ -7,\ -4,\ 5,\ -3,\ -7,\ -2\ ))$ *be linearly independent matrices in* $\mathcal{D}_7$. *Our aim is to find the best coapproximation(s) to any given* T *out of* $\mathbb{Y} = span\{A_1, A_2, A_3\}$. *In view of the Theorem 2.6, we proceed in the following steps.*

Step 1 : *For* $i \in \{1, 2, \ldots, 7\}$, *the* i*-th components are respectively*

$$(7,1,3), (-5,3,-7), (2,4,-4), (6,3,5), (-7,-1,-3), (-5,3,-7), (1,2,-2).$$

Step 2 : $P_1^+ = \{1\}, P_1^- = \{5\}$; $P_2^+ = \{2,6\}, P_2^- = \emptyset$; $P_3^+ = \{3\}, P_3^- = \emptyset$; $P_4^+ = \{4\}, P_4^- = \emptyset$; $P_5^+ = \{5\}, P_5^- = \{1\}$; $P_6^+ = \{2,6\}, P_6^- = \emptyset$; $P_7^+ = \{7\}, P_7^- = \emptyset$, *respectively, where* P_i^+ *and* P_i^- *are the positively associated set and the negatively associated set of the* i*-th component, respectively, for all* $i \in \{1, 2, \ldots, 7\}$.

Step 3 : *Here the nonequivalent components satisfying the* $*$*-Property may be taken as the* 1*-st component, the* 2*-nd component, the* 3*-rd component and the* 4*-th component.*

Step 4 : *In this final step, we consider a given* $T \in \mathcal{M}_7$ *and apply Theorem 2.6 to obtain the best coapproximation to* T *out of* $\mathbb{Y}$. *In order to illustrate the various possibilities arising in the best coapproximation problem in* $\mathcal{D}_7$, *it suffices to consider the following three particular cases.*

Case 1 : *Let* $T_1 \in \mathcal{M}_7$ *be given by* $T_1 = (b_{ij})_{1 \le i,j \le 7}$, *where* $b_{11} = 2,\ b_{15} = 4,\ b_{22} = 1,\ b_{26} = 3,\ b_{33} = 4,\ b_{44} = 1,\ b_{51} = -7,\ b_{55} = -2,\ b_{62} = 2,\ b_{66} = 1$ *and the other* b_{ij}*'s can be chosen arbitrarily.*

Therefore, ${}^1T_{1*} = \begin{pmatrix} 2 & 4 \\ 7 & 2 \end{pmatrix}$, ${}^2T_{1*} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$, ${}^3T_{1*} = (4)$, ${}^4T_{1*} = (1)$.

Then from Theorem 2.6, $\sum_{i=1}^{3} \alpha_i A_i$ *is a best coapproximation to* T_1 *out of* $\mathbb{Y}$ *if and only if* $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ *satisfies the following relations:*

$$\begin{aligned} 7\alpha_1 + \ \alpha_2 + \ 3\alpha_3 \in \ W\left({}^1T_{1*}\right) &= [-7/2,\ 15/2] \\ -5\alpha_1 + \ 3\alpha_2 - \ 7\alpha_3 \in \ W\left({}^2T_{1*}\right) &= [-3/2,\ 7/2] \\ 2\alpha_1 + \ 4\alpha_2 - \ 4\alpha_3 \in \ W\left({}^3T_{1*}\right) &= \{4\} \end{aligned}$$

$$6\alpha_1 + 3\alpha_2 - 5\alpha_3 \in W\left({}^4T_{1*}\right) = \{1\}.$$

Since there are infinitely many $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ satisfying the above relations, there are infinitely many best coapproximation to T_1 out of $\mathbb{Y}$. Moreover,

$$\mathcal{R}_{\mathbb{Y}}(T_1) = \{((x,\ 4-x,\ 4,\ 1,\ -x,\ 4-x\ ,2))\ :\ 1/2 \leq x \leq 11/2\}.$$

Case 2 : Let $T_2 \in \mathcal{M}_7$ be given by $T_1 = (c_{ij})_{1\leq i,j\leq 7}$, where $c_{11} = 3,\ c_{15} = -5,\ c_{22} = 1,\ c_{26} = 3,\ c_{33} = 4,\ c_{44} = 1,\ c_{51} = -5,\ c_{55} = -3,\ c_{62} = 2,\ c_{66} = 1$ and the other c_{ij}'s can be chosen arbitrarily.

Therefore, $^1T_{2*} = \begin{pmatrix} 3 & -5 \\ 5 & 3 \end{pmatrix},\ \ ^2T_{2*} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix},\ ^3T_{2*} = (4),\ ^4T_{2*} = (1).$
Then from Theorem 2.6,

$$\begin{aligned} 7\alpha_1 + \alpha_2 + 3\alpha_3 \in W\left({}^1T_{2*}\right) &= \{3\} \\ -5\alpha_1 + 3\alpha_2 - 7\alpha_3 \in W\left({}^2T_{2*}\right) &= [-3/2,\ 7/2] \\ 2\alpha_1 + 4\alpha_2 - 4\alpha_3 \in W\left({}^3T_{2*}\right) &= \{4\} \\ 6\alpha_1 + 3\alpha_2 - 5\alpha_3 \in W\left({}^4T_{2*}\right) &= \{1\}. \end{aligned}$$

Since there exist unique $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ satisfying the above relations, the best coapproximation to T out of $\mathbb{Y}$ is unique. Moreover,

$$\mathcal{R}_{\mathbb{Y}}(T_2) = \{((3,\ 1,\ 4,\ 1,\ -3,\ 1\ ,2))\}.$$

Case 3 : Let $T_3 \in \mathcal{M}_7$ be given by $T_3 = (d_{ij})_{1\leq i,j\leq 7}$, where $d_{11} = 14,\ d_{15} = 1,\ d_{22} = 1,\ d_{26} = 3,\ d_{33} = 4,\ d_{44} = 1,\ d_{51} = 1,\ d_{55} = -14,\ d_{62} = 2,\ d_{66} = 1$ and the other d_{ij}'s can be chosen arbitrarily.

Therefore, $^1T_{3*} = \begin{pmatrix} 14 & 1 \\ -1 & 14 \end{pmatrix},\ \ ^2T_{3*} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix},\ ^3T_{3*} = (4),\ ^4T_{3*} = (1).$
Then from Theorem 2.6,

$$\begin{aligned} 7\alpha_1 + \alpha_2 + 3\alpha_3 \in W\left({}^1T_{3*}\right) &= \{14\} \\ -5\alpha_1 + 3\alpha_2 - 7\alpha_3 \in W\left({}^2T_{3*}\right) &= [-3/2,\ 7/2] \\ 2\alpha_1 + 4\alpha_2 - 4\alpha_3 \in W\left({}^3T_{3*}\right) &= \{4\} \\ 6\alpha_1 + 3\alpha_2 - 5\alpha_3 \in W\left({}^4T_{3*}\right) &= \{1\}. \end{aligned}$$

Since there exists no such $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ satisfying the above relations, it follows that

$$\mathcal{R}_{\mathbb{Y}}(T_3) = \emptyset.$$

2.3.1 Characterization of coproximinal subspaces

We now obtain a characterization of the coproximinal subspaces of $\mathcal{D}_n$ in terms of the $*$-Property.

Theorem 2.10. *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be linearly independent in $\mathcal{D}_n$, where $A_k = ((a_{11}^k, a_{22}^k, \ldots, a_{nn}^k))$, for each $1 \leq k \leq m$. Then span $\{A_1, A_2, \ldots, A_m\}$ is a coproximinal subspace of $\mathcal{M}_n$ if and only if there exist exactly m number of nonequivalent components satisfying the $*$-Property.*

Proof. Let us first prove the sufficient part of the theorem. Let the j_1-th, j_2-th, $\ldots$, j_m-th components be chosen as the nonequivalent m number of components satisfying the $*$-Property. Let us consider $C \in \mathcal{M}_m$ given by $C = (c_{st})_{1 \leq s,t \leq m}$ such that $c_{st} = a_{j_s j_s}^t$, where $\left(a_{j_s j_s}^1, a_{j_s j_s}^2, \ldots, a_{j_s j_s}^m\right)$ is the j_s-th component. We claim that $rank(C) = m$. Suppose on the contrary $rank(C) < m$. Let $Y_1 = span\{\left(a_{j_s j_s}^1, a_{j_s j_s}^2, \ldots, a_{j_s j_s}^m\right) : 1 \leq s \leq m\}$ and let $Y_2 = span\{\left(a_{ii}^1, a_{ii}^2, \ldots, a_{ii}^m\right) : 1 \leq i \leq n\}$. Clearly, $Y_1 \subsetneq Y_2 = \mathbb{R}^m$, which implies that $Y_2^\perp \subsetneq Y_1^\perp$. Therefore, there exists $(\gamma_1, \gamma_2, \ldots, \gamma_m) \in Y_1^\perp \setminus Y_2^\perp$ such that $|\sum_{k=1}^m \gamma_k a_{ii}^k| > \max\{|\sum_{k=1}^m \gamma_k a_{j_s j_s}^k| : 1 \leq s \leq m\} = 0$, for some $i \notin \{j_1, j_2, \ldots, j_m\}$. Following Theorem 2.1, there exists an i-th component, which is nonequivalent to the j_1-th, j_2-th, $\ldots$, j_m-th components, that satisfies the $*$-Property. This contradiction establishes our claim. Therefore, C is invertible and hence onto. So, for any $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$, there always exist $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ such that $a_{j_s j_s}^1 \alpha_1 + a_{j_s j_s}^2 \alpha_2 + \ldots + a_{j_s j_s}^m \alpha_m = \beta_i$, for all $s \in \{1, 2, \ldots, m\}$. Noting that for any $T \in \mathcal{M}_n$, $W({}^{j_s}T_*) \subset \mathbb{R}$, therefore we conclude that

$$a_{j_s j_s}^1 \alpha_1 + a_{j_s j_s}^2 \alpha_2 + \ldots + a_{j_s j_s}^m \alpha_m \in W\left({}^{j_s}T_*\right), \tag{2.3}$$

for all $s \in \{1, 2, \ldots, m\}$. Following Theorem 2.6, it is now evident that $\sum_{k=1}^m \alpha_k A_k$ is the best coapproximation to T out of *span* $\{A_1, A_2, \ldots, A_m\}$. This shows that *span* $\{A_1, A_2, \ldots, A_m\}$ is a coproximinal subspace of $\mathcal{M}_n$.

Let us now prove the necessary part of the theorem. Suppose that the j_1-th, j_2-th, $\ldots$, j_p-th nonequivalent components satisfy the $*$-Property. Then from Lemma 2.2, we get that $p \geq m$. Let us now take the $p \times m$ matrix $D = (d_{st})_{1 \leq s \leq p,\ 1 \leq t \leq m}$ such that $d_{st} = a_{j_s j_s}^t$, where $\left(a_{j_s j_s}^1, a_{j_s j_s}^2, \ldots, a_{j_s j_s}^m\right)$ is the j_s-th component. Let $T_D \in \mathbb{L}(\mathbb{H}_1, \mathbb{H}_2)$ be the linear operator corresponding to the matrix D with respect to the standard ordered bases of $\mathbb{H}_1, \mathbb{H}_2$, where

$\mathbb{H}_1 = \mathbb{R}^m$ and $\mathbb{H}_2 = \mathbb{R}^p$. Since *span* $\{A_1, A_2, \ldots, A_m\}$ is a coproximinal subspace of $\mathcal{M}_n$, for $T \in \mathcal{M}_n$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ satisfying the following relations:

$$a^1_{j_sj_s}\alpha_1 + a^2_{j_sj_s}\alpha_2 + \ldots + a^m_{j_sj_s}\alpha_m \in W\left({}^{j_s}T_*\right), \tag{2.4}$$

for all $s \in \{1, 2, \ldots, p\}$. We now claim that T_D is onto. Let $\beta = (\beta_1, \beta_2, \ldots, \beta_p) \in \mathbb{R}^p$. We note that for any $T = (b_{ij})_{1\leq i,j\leq n}$, ${}^{j_s}T_*$ is a $h \times h$ matrix whose entries are $(\pm b_{ij})$ depending on $P^+_{j_s}$ and $P^-_{j_s}$, where $|P^+_{j_s} \cup P^-_{j_s}| = h$. So we can choose T suitably so that $W({}^{j_s}T_*) = \{\beta_s\}$ for each $s \in \{1, 2, \ldots, p\}$. This shows that for each $s \in \{1, 2, \ldots, p\}$, we get

$$a^1_{j_sj_s}\alpha_1 + a^2_{j_sj_s}\alpha_2 + \ldots + a^m_{j_sj_s}\alpha_m = \beta_s$$

and so $T_D(\alpha) = \beta$, where $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_m) \in \mathbb{R}^m$. Thus T_D is onto and therefore, $m \geq p$. This along with Lemma 2.2 completes the proof. □

2.3.2 Characterization of co-Chebyshev subspaces

We now obtain a characterization of the co-Chebyshev subspaces of $\mathcal{D}_n$ in terms of the $*$-Property.

Theorem 2.11. *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be linearly independent in $\mathcal{D}_n$, where $A_k = ((a^k_{11}, a^k_{22}, \ldots, a^k_{nn}))$, for each $1 \leq k \leq m$. Suppose that the i_1-th, i_2-th, …, i_p-th nonequivalent components satisfy the $*$-Property. Then span$\{A_1, A_2, \ldots, A_m\}$ is a co-Chebyshev subspace of $\mathcal{M}_n$ if and only if $p = m$ and $|P^+_{i_s} \cup P^-_{i_s}| = 1$ for all $s \in \{1, 2, \ldots, p\}$.*

Proof. We first prove the sufficient part of the theorem. Since $p = m$, we note from Theorem 2.10 that *span*$\{A_1, A_2, \ldots, A_m\}$ is a coproximinal subspace of $\mathcal{M}_n$. Therefore, for any given $T \in \mathcal{M}_n$, there exist $\alpha_1, \alpha_2 \ldots, \alpha_m \in \mathbb{R}$ satisfying the following relations:

$$a^1_{i_si_s}\alpha_1 + a^2_{i_si_s}\alpha_2 + \ldots + a^m_{i_si_s}\alpha_m \in W\left({}^{i_s}T_*\right), \tag{2.5}$$

for all $s \in \{1, 2, \ldots, m\}$. Since $|P^+_{i_s} \cup P^-_{i_s}| = 1$, it follows that ${}^{i_s}T_*$ is of order 1. Moreover, $W\left({}^{i_s}T_*\right) = (b_{i_si_s})$, for every $s \in \{1, 2, \ldots, m\}$, where $T = (b_{ij})_{1\leq i,j\leq n}$. Therefore, relations (2.5) represent a system of linear equation with coefficient matrix $C = (c_{st})_{1\leq s,t\leq m}$, where $c_{st} = a^t_{i_si_s}$. Following the arguments given in the proof of Theorem 2.10, we conclude that C is invertible. Hence for any given $T \in \mathcal{M}_n$, there exists a unique $(\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m$ satisfying the relations (2.5). Therefore, *span*$\{A_1, A_2, \ldots, A_m\}$ is a co-Chebyshev subspace of $\mathcal{M}_n$.

We now prove the necessary part of the theorem. Let us assume that $span\{A_1, A_2, \ldots, A_m\}$ is a co-Chebyshev subspace of $\mathcal{M}_n$. In particular, $span\{A_1, A_2, \ldots, A_m\}$ is a coproximinal subspace of $\mathcal{M}_n$. Therefore, from Theorem 2.10, we get $p = m$. Suppose on the contrary $|P^+_{i_s} \cup P^-_{i_s}| = k_s > 1$, for some $s \in \{1, 2, \ldots, m\}$. Therefore, ${}^{i_s}T_*$ is of order k_s. Let us consider $Q = (q_{rt})_{1 \leq r \leq p, 1 \leq t \leq m}$, where $q_{rt} = a^t_{i_r i_r}$. Since $p = m$, following the arguments given in the proof of Theorem 2.10, Q is invertible. Now, for any two scalars c and d, where $c \neq d$, we can choose a suitable $T \in \mathcal{M}_n$ such that $c, d \in W\left({}^{i_s}T_*\right)$. Therefore, we can conclude that there exist at least two different sets of $(\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m$ satisfying the relations:

$$a^1_{i_l i_l}\alpha_1 + a^2_{i_l i_l}\alpha_2 + \ldots + a^m_{i_l i_l}\alpha_m \in W\left({}^{i_l}T_*\right),$$

for all $l \in \{1, 2, \ldots, m\}$. This contradicts that $span\{A_1, A_2, \ldots, A_m\}$ is a co-Chebyshev subspace of $\mathcal{M}_n$. Hence the theorem. □

As an immediate application of the above theorem, we record the following interesting observation.

Corollary 2.1. *$\mathcal{D}_n$ is a co-Chebyshev subspace of $\mathcal{M}_n$.*

Proof. Clearly, $\{E_k : 1 \leq k \leq n\}$ is a basis of $\mathcal{D}_n$, where $E_k = (e^k_{ij})_{1 \leq i,j \leq n}$ is given by

$$\begin{aligned} e^k_{ij} &= 1, \text{ whenever } i = j = k \\ &= 0, \text{ otherwise.} \end{aligned}$$

It is trivial to observe that for all $1 \leq i \leq n$, the i-th component satisfies the $*$-Property and $|P^+_i \cup P^-_i| = 1$. Therefore, the desired result follows directly from Theorem 2.11. □

2.4 Best coapproximation in ℓ^n_∞

As another important application of the theories developed in the present chapter, it is possible to characterize the best coapproximation problem in the setting of ℓ^n_∞, for any given $n \in \mathbb{N}$. This in turn is equivalent to the following optimization problem:

Problem: Let $a_{ij}, \alpha_k \in \mathbb{R}$ be fixed, where $1 \leq i \leq m, 1 \leq j, k \leq n$. Find a necessary and sufficient condition for the existence of $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$ such that for any $c_1, c_2, \ldots, c_m \in \mathbb{R}$, the following inequality holds true:

$$\max\left\{|\alpha_k - \sum_{i=1}^{m} c_i a_{ik}| : 1 \leq k \leq n\right\} \geq \max\left\{|\sum_{i=1}^{m} \beta_i a_{ik} - \sum_{i=1}^{m} c_i a_{ik}| : 1 \leq k \leq n\right\}.$$

Moreover, in case existence is guaranteed, find all such $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$.

We emphasize that the above problem is not entirely trivial, most notably because the existence of a desired solution is not a priori guaranteed. However, it is possible to completely solve the problem (from both theoretical and computational perspectives), by applying the methodology already developed in this chapter. It is well-known that ℓ_∞^n (endowed with its usual maximum norm) is isometrically isomorphic to $\mathcal{D}_n$ endowed with the usual operator norm. Indeed, the natural choice map $\Psi : \ell_\infty^n \longrightarrow \mathcal{D}_n$, taking $(a_1, a_2, \ldots, a_n) \in \ell_\infty^n$ to $((a_1, a_2, \ldots, a_n)) \in \mathcal{D}_n$ is easily seen to be the desired isometric isomorphism. This connection allows us to obtain an algorithmic approach to the best coapproximation problem in any given subspace $\mathbb{Y}$ of ℓ_∞^n via the methods already developed to treat the corresponding best coapproximation problem in the subspace $\Psi(\mathbb{Y})$ of $\mathcal{D}_n$. It should be noted in this context that our theory essentially translates into characterizing the best coapproximation(s) to any given $T \in \mathcal{M}_n$ out of any given subspace of ℓ_∞^n, and therefore, the best coapproximation problem in subspaces of ℓ_∞^n is only a particular case of the results developed so far. To illustrate this further, we make note of the following two remarks pertaining to the best coapproximation problem in ℓ_∞^n :

- ℓ_∞^n is a coproximinal subspace of $\mathcal{M}_n$ for each $n \in \mathbb{N}$. This is simply a reformulation of Corollary 2.1.
- By using the concept of the $*$-Property, and the above mentioned isometric isomorphism $\Psi : \ell_\infty^n \longrightarrow \mathcal{D}_n$, it is quite straightforward to construct subspaces of ℓ_∞^n which are (not) coproximinal. Indeed, in light of the theories developed in this chapter, any such construction essentially reduces to controlling the number (nonequivalent) of i-th components satisfying the $*$-Property, for $1 \leq i \leq n$. As an explicit example, it can be readily verified that $\mathbb{Y}_1$ is a coproximinal subspace of l_∞^7, whereas $\mathbb{Y}_2$ is not, where

$$\mathbb{Y}_1 = span\left\{(6,1,4,3,3,1,1),(2,5,2,3,1,5,1),(4,3,8,6,2,3,2),(2,1,4,9,1,1,3)\right\},$$

$$\mathbb{Y}_2 = span\left\{(2,-5,3,1,-2,-5,2),(-4,2,2,-2,-4,2,-4)\right\}.$$

In view of our treatment of the theory of coapproximations in ℓ_∞^n spaces, it seems appropriate to end the present chapter with the following concluding remark:

Remark 2.12. *The theory of best coapproximations in Banach spaces remains a much less exposed area of research, especially from a computational point of view, in comparison to the theory of best approximations. Certainly, this is in part due to the inherently complicated non-linear nature of the best coapproximation problem and the difficulty of the corresponding computations involved in the process. In this context, the reader is encouraged to look up the literature, including [33, 36]. Our main focus in this chapter is to illustrate the following principle in the setting of* ℓ_∞^n *(or, more generally, for matrices in* $\mathcal{M}_n$ *out of subspaces of* $\mathcal{D}_n$*):*

It is possible to essentially reduce the much harder "best coapproximation problem" to the well-known and way more simpler "existence and uniqueness problem corresponding to a particular system of linear problems", by applying the concept of orthogonality.

Indeed, using the methodology developed so far, it is now very easy to explicitly produce examples of coproximinal and co-Chebyshev subspaces in the setting of ℓ_∞^n*. Therefore, in light of the above fact, a natural query would be to test the validity of such a nicety, in the setting of classical Banach spaces other than* ℓ_∞^n*.*

CHAPTER 3

ON BEST COAPPROXIMATION PROBLEM IN ℓ_1^n

3.1 Introduction

As seen in the previous chapter, finding the best coapproximation in Banach spaces is a challenging problem, particularly from a computational standpoint. In that chapter, we focused on a complete characterization of the problem within the space of diagonal matrices, which enabled a computational resolution in certain subspaces of ℓ_∞^n. A natural and compelling direction for extending this investigation is to explore the best coapproximation problem in subspaces of the dual space, ℓ_1^n. This space, with its distinct geometric and duality properties, offers a contrasting yet complementary setting to ℓ_∞^n. As an application of our methodology, we provide a complete identification of coproximinal and co-Chebyshev subspaces of ℓ_1^n. It is time to mention the basic terminologies and the notations to be used throughout the chapter.

We use the symbols $\mathbb{X}, \mathbb{Y}$ to denote real Banach spaces, unless stated otherwise. The usual notations $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq 1\}$ and $S_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| = 1\}$ are used to denote the unit

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ball and the unit sphere of $\mathbb{X}$, respectively. An element $x \in B_{\mathbb{X}}$ is said to be an extreme point of the unit ball if $x = (1-t)y + tz$, for some $t \in (0,1)$ and for some $y, z \in B_{\mathbb{X}}$, then $x = y = z$. The collection of all extreme points of the unit ball $B_{\mathbb{X}}$ is denoted by $Ext(B_{\mathbb{X}})$. The dual of a Banach space $\mathbb{X}$ is denoted by $\mathbb{X}^*$. Given any $f \in \mathbb{X}^*$, M_f denotes the norm attainment set of f, i.e., $M_f := \{x \in S_{\mathbb{X}} : |f(x)| = \|f\|\}$. We note that $M_f \neq \emptyset$, whenever $\mathbb{X}$ is reflexive. Given any $m \times n$ matrix A, A^t denotes the transpose of A. Let $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ be denoted as the Banach space of all bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$, endowed with the usual operator norm. For given any $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, the kernel of T, denoted by $\ker T$, is defined as $\ker T := \{x \in \mathbb{X} : Tx = \theta \in \mathbb{Y}\}$. Accordingly, the kernel of $f \in \mathbb{X}^*$ is denoted by $\ker f$, i.e., $\ker f = \{x \in \mathbb{X} : f(x) = 0\}$. Given $x \in \mathbb{X}$ and a subspace $\mathbb{Y}$ of $\mathbb{X}$, we denote by $\mathcal{R}_{\mathbb{Y}}(x)$ the set of all best coapproximations to x out of $\mathbb{Y}$. We also define *dom* $\mathcal{R}_{\mathbb{Y}}$ as the collection of all such $x \in \mathbb{X}$ such that $\mathcal{R}_{\mathbb{Y}}(x) \neq \emptyset$.

The study of best coapproximation has an immediate connection to *the concept of Birkhoff-James orthogonality.* Following the pioneering articles [2, 20], given any two elements x, y in a Banach space $\mathbb{X}$, we say that *x is Birkhoff-James orthogonal to y*, written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$. The said connection can be stated (and verified in a rather straightforward manner) in terms of Birkhoff-James orthogonality as follows: Given a subspace $\mathbb{Y}$ of a Banach space $\mathbb{X}$ and an element $x \in \mathbb{X}$, $y_0 \in \mathbb{Y}$ is a best coapproximation to x out of $\mathbb{Y}$ if and only if $\mathbb{Y} \perp_B (x - y_0)$ i.e., $y \perp_B (x - y_0)$ for all $y \in \mathbb{Y}$.

In this chapter, our main objective is to completely solve the problem of finding the best coapproximation(s) to a given element in ℓ_1^n out of a given subspace $\mathbb{Y}$ of ℓ_1^n, provided the best coapproximation(s) exist. As mentioned before, it is known that given a subspace $\mathbb{Y}$ of ℓ_1^n and $x \notin \mathbb{Y}$, y_0 is a best coapproximation to x out of $\mathbb{Y}$ if and only if there exists a norm-1 projection from *span* $\{x, \mathbb{Y}\}$ to $\mathbb{Y}$. However, to the best of our knowledge, there is no method available to explicitly find these norm-1 projections. We present a computationally effective solution to this problem, resulting in an algorithmic approach to the best coapproximation problem in ℓ_1^n. It also allows us to discuss the existence of best coapproximations in the said setting. We also completely identify the coproximinal and co-Chebyshev subspaces of ℓ_1^n.

3.2 Preliminaries

We introduce the following definition for computational advantages.

Definition 3.1. *Let $\mathcal{A} = \{\widetilde{a_1}, \widetilde{a_2}, \ldots, \widetilde{a_m}\}$ be a set of linearly independent elements in ℓ_1^n, where $\widetilde{a_k} = (a_1^k, a_2^k, \ldots, a_n^k)$, for each $1 \leq k \leq m$. Considering $\widetilde{a_1}, \widetilde{a_2}, \ldots, \widetilde{a_m}$ as column vectors, we form the $n \times m$ matrix $A = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$, where $a_{ij} = a_i^j$.*

(i) *For each* $i \in \{1, 2, \ldots, n\}$, *the* i*-th component of* $\mathcal{A}$ *is defined as the* i*-th row of* A, *i.e.,* $\left(a_i^1, a_i^2, \ldots, a_i^m\right)$. *Whenever the context is clear, we simply refer to the* i*-th component of* $\mathcal{A}$ *as the* i*-th component.*

(ii) *The* i*-th component and the* j*-th component are said to be* equivalent *if*

$$\left(a_i^1, a_i^2, \ldots, a_i^m\right) = c\left(a_j^1, a_j^2, \ldots, a_j^m\right), \quad c(\neq 0) \in \mathbb{R}.$$

(iii) *The* zero set $\mathcal{Z}_{\mathcal{A}}$ *of* $\mathcal{A}$ *is defined as*

$$\mathcal{Z}_{\mathcal{A}} = \Big\{i \in \{1, 2, \ldots, n\} : \left(a_i^1, a_i^2, \ldots, a_i^m\right) = (0, 0, \ldots, 0)\Big\}.$$

In the following proposition, we show the basis invariance of *the equivalent components and the zero set.*

Proposition 3.1. *Let* $\mathbb{Y}$ *be a subspace of* ℓ_1^n *and let* $\mathcal{A} = \{\widetilde{a_1}, \widetilde{a_2}, \ldots, \widetilde{a_m}\}, \mathcal{B} = \{\widetilde{b_1}, \widetilde{b_2}, \ldots, \widetilde{b_m}\}$ *be two bases of* $\mathbb{Y}$, *where* $\widetilde{a_k} = (a_1^k, a_2^k, \ldots, a_n^k)$ *and* $\widetilde{b_k} = (b_1^k, b_2^k, \ldots, b_n^k)$, *for any* $1 \leq k \leq m$. *Then*

(i) *the* i*-th and* j*-th component of* $\mathcal{A}$ *are equivalent if and only if the* i*-th and* j*-th component of* $\mathcal{B}$ *are equivalent.*

(ii) $\mathcal{Z}_{\mathcal{A}} = \mathcal{Z}_{\mathcal{B}}$.

Proof. (i) Consider the two matrices A and B as constructed in Definition 3.1. Since $\mathcal{A}$ and $\mathcal{B}$ are two bases of $\mathbb{Y}$, there exists an invertible matrix $C = (c_{ij})_{1 \leq i,j \leq m}$ such that $B = AC$, where $b_{ij} = \sum_{k=1}^m a_{ik} c_{kj}$, for any $1 \leq i \leq n, 1 \leq j \leq m$. The j-th components of $\mathcal{B}$,

$$(b_j^1, b_j^2, \ldots, b_j^m) = \left(\sum_{k=1}^m c_{k1} a_j^k, \sum_{k=1}^m c_{k2} a_j^k, \ldots, \sum_{k=1}^m c_{km} a_j^k\right). \tag{3.1}$$

Suppose that the i-th and the j-th components of $\mathcal{A}$ are equivalent. Then $(a_j^1, a_j^2, \ldots, a_j^m) = c(a_i^1, a_i^2, \ldots, a_i^m)$, for some $c \in \mathbb{R}$. Therefore,

$$(b_j^1, b_j^2, \ldots, b_j^m) = c\left(\sum_{k=1}^m c_{k1} a_i^k, \sum_{k=1}^m c_{k2} a_i^k, \ldots, \sum_{k=1}^m c_{km} a_i^k\right) = c(b_i^1, b_i^2, \ldots, b_i^m).$$

This implies that the i-th and the j-th components of $\mathcal{B}$ are equivalent. By a similar argument, we can easily obtain the converse result.

(ii) Follows immediately from equation (1). □

We next obtain a characterization of best coapproximation(s) in finite-dimensional subspaces of the dual of a reflexive Banach space. This simple observation will play an important role in finding the best coapproximation in the subspaces of ℓ_1^n.

Theorem 3.1. *Let $\mathbb{X}$ be a reflexive Banach space and let $g_1, g_2, \ldots, g_m \in \mathbb{X}^*$ be linearly independent. Given any $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$, $\sum_{k=1}^m \alpha_k g_k$ is a best coapproximation to $f \in \mathbb{X}^*$ out of span$\{g_1, g_2, \ldots, g_m\}$ if and only if given any $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$,*

$$M_{\sum_{k=1}^m \beta_k g_k} \cap \ker(f - \sum_{k=1}^m \alpha_k g_k) \neq \emptyset$$

Proof. It follows from the definitions of Birkhoff-James orthogonality and best coapproximation that $\sum_{k=1}^m \alpha_k g_k$ is a best coapproximation to f out of *span*$\{g_1, g_2, \ldots, g_m\}$ if and only if $g \perp_B (f - \sum_{k=1}^m \alpha_k g_k)$, for all $g \in$ *span* $\{g_1, g_2, \ldots, g_m\}$. Clearly, this is equivalent to the following:

$$\sum_{k=1}^m \beta_k g_k \perp_B (f - \sum_{k=1}^m \alpha_k g_k) \ \forall\ \beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}.$$

Now applying [45, Th. 3.2], we conclude that the above condition is equivalent to

$$M_{\sum_{k=1}^m \beta_k g_k} \cap \ker(f - \sum_{k=1}^m \alpha_k g_k) \neq \emptyset.$$

This completes the proof of the theorem. □

In this chapter, we address the problem by classifying the subspaces of ℓ_1^n based on whether the associated zero set, as defined above, is empty or non-empty. In the case where the zero set is empty, we provide a complete characterization by introducing a norming property of the subspace. When the zero set is non-empty, we approach the problem by truncating it to a subspace in which the zero set becomes empty. To proceed with this approach, we now introduce a definition that will be instrumental in solving the problem in cases where the zero set is non-empty.

Definition 3.2. *Let $\mathbb{Y}$ be a subspace of ℓ_1^n and let $\mathcal{A} = \{\widetilde{a_1}, \widetilde{a_2}, \ldots, \widetilde{a_m}\}$ be a basis of $\mathbb{Y}$ with $\mathcal{Z}_\mathcal{A} \neq \emptyset$. Suppose that $|\mathcal{Z}_\mathcal{A}| = r > 0$ and $n - r = k$. Without loss of generality we assume that $\{1, 2, \ldots, n\} \setminus \mathcal{Z}_\mathcal{A} = \{1, 2, \ldots, k\}$.*

(i) We define a linear transformation ρ from ℓ_1^n to ℓ_1^n by

$$\rho(b_1, b_2, \ldots, b_n) = (c_1, c_2, \ldots, c_n),$$

where $c_i = b_i$, *for any* $i \notin \mathcal{Z}_{\mathcal{A}}$ *and* $c_i = 0$, *for any* $i \in \mathcal{Z}_{\mathcal{A}}$.

(ii) We define a linear transformation σ *from* ℓ_1^n *to* ℓ_1^k *by*

$$\sigma(b_1, b_2, \ldots, b_n) = (b_1, b_2, \ldots, b_k).$$

(iii) For a given $\widetilde{b} = (b_1, b_2, \ldots, b_n) \in \ell_1^n$, *we introduce a set* $\mathcal{P}_{\widetilde{b}} \subset \ell_1^n$, *defined as*

$$\mathcal{P}_{\widetilde{b}} := \Big\{ \widetilde{y} = (y_1, y_2, \ldots, y_n) \in \ell_1^n : y_i = b_i \ \forall j \notin \mathcal{Z}_{\mathcal{A}} \Big\}.$$

We note the following simple but useful properties in the form of a proposition.

Proposition 3.2. *Let* $\mathbb{Y}$ *be a subspace of* ℓ_1^n *and let* $\mathcal{A} = \{\widetilde{a_1}, \widetilde{a_2}, \ldots, \widetilde{a_m}\}$ *be a basis of* $\mathbb{Y}$ *with* $\mathcal{Z}_{\mathcal{A}} \neq \emptyset$. *Suppose that* $|\mathcal{Z}_{\mathcal{A}}| = r > 0$ *and* $n - r = k$. *For any* $f \in (\ell_\infty^n)^*$,

(i) $\rho(\widetilde{a_i}) = \widetilde{a_i}$ *and* $\|\rho(\widetilde{b})\| \leq \|\widetilde{b}\|$.

(ii) $\|\sigma(\widetilde{b})\| \leq \|\widetilde{b}\|$ *and* $\|\sigma(\widetilde{a_i})\| = \|\widetilde{a_i}\|$.

(iii) $\sigma(\rho(\widetilde{b})) = \sigma(\widetilde{b})$ *and* $\rho(\rho(\widetilde{b})) = \rho(\widetilde{b})$.

(iv) for any $\widetilde{y} \in \mathcal{P}_{\widetilde{b}}$, $\rho(\widetilde{b}) = \rho(\widetilde{y})$.

(v) $\mathcal{Z}_{\sigma(\mathcal{A})} = \emptyset$, *where* $\sigma(\mathcal{A}) = \{\sigma(\widetilde{a_1}), \sigma(\widetilde{a_2}), \ldots, \sigma(\widetilde{a_m})\}$.

In the following section, we focus on subspaces $\mathbb{Y}$ of ℓ_1^n spanned by the basis $\mathcal{A}$ with $\mathcal{Z}_{\mathcal{A}} = \emptyset$.

3.3 Norming set

We begin this section by noting that there exists a canonical isometric isomorphism ψ between ℓ_1^n and $(\ell_\infty^n)^*$, defined as $\psi(a_1, a_2, \ldots, a_n) = g$, where $g : \ell_\infty^n \to \mathbb{R}$ is given as :

$$g(\alpha_1 e_1 + \alpha_2 e_2 + \ldots \alpha_n e_n) = \alpha_1 a_1 + \alpha_2 a_2 + \ldots \alpha_n a_n,$$

$\{e_1, e_2, \ldots, e_n\}$ being the standard ordered basis of $\mathbb{R}^n$. Thus, given a subspace $\mathbb{Y}$ of ℓ_1^n and an element $x \notin \mathbb{Y}$, the problem of finding best coapproximation to x out of $\mathbb{Y}$ is equivalent to the problem of finding the same to $\psi(x)$ out of the subspace $\psi(\mathbb{Y})$ in $(\ell_\infty^n)^*$. This observation will be used as and when required.

Now it is time to introduce the following norming property of a subspace in ℓ_1^n, which helps us to to solve the best coapproximation problem.

Definition 3.3. *A set S in a Banach space is said to be symmetric if $x \in S$ implies $-x \in S$.*

Definition 3.4. *Let $\mathbb{Y}$ be a subspace of ℓ_1^n. A symmetric set $\mathcal{N}$ is said to be a norming set of $\mathbb{Y}$ if $\big(M_g \cap Ext(B_{\ell_\infty^n})\big) \cap \mathcal{N} \neq \emptyset$, for each $g \in \psi(\mathbb{Y})$. A norming set $\mathcal{N}$ is said to be a minimal norming set of $\mathbb{V}$ if for some norming set $\mathcal{M}$ of $\mathbb{Y}$, $\mathcal{M} \subset \mathcal{N}$ implies that $\mathcal{M} = \mathcal{N}$.*

Observe that $M_g \cap Ext(B_{\ell_\infty^n}) \neq \emptyset$, for each $g \in \psi(\mathbb{Y})$. Clearly, the minimal norming set may not be unique. Let $g \in (\ell_\infty^n)^*$. Then for any $x = (x_1, x_2, \ldots, x_n) \in \ell_\infty^n$, $g(x) = \sum_{i=1}^n g(e_i)x_i$. The following result ensures the existence of the minimal norming set of a subspace $\mathbb{Y}$ of ℓ_1^n.

Theorem 3.2. *Let $\mathbb{Y}$ be a subspace of ℓ_1^n and let $\mathcal{A} = \{\widetilde{a_1}, \widetilde{a_2}, \ldots, \widetilde{a_m}\}$ be a basis of $\mathbb{Y}$ with $\mathcal{Z}_{\mathcal{A}} = \emptyset$. Then there exists a unique minimal norming set of $\mathbb{Y}$.*

Proof. Suppose that $\psi(\widetilde{a_i}) = g_i$, for any $1 \leq i \leq m$. Since $\mathcal{Z}_{\mathcal{A}} = \emptyset$, we observe that

$$\Big\{e_1, e_2, \ldots, e_n\Big\} \bigcap \Big(\bigcap_{j=1}^{m} \ker g_j\Big) = \emptyset.$$

Any element g of $\psi(\mathbb{Y})$ is of the form $g = \sum_{k=1}^m \beta_k g_k$, where $(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{R}^m$. Moreover, we note that $g_k(e_i) = a_i^k$, for any $1 \leq k \leq m, 1 \leq i \leq n$. We will prove the theorem in the following four steps.

Step 1 : We express $\mathbb{R}^m$ as the union of finitely many hyperplanes and open sets which are relevant to our purpose. For each $i = 1, 2, \ldots, n$, we consider the hyperplane H_i of $\mathbb{R}^m$, given by

$$H_i = \Big\{(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{R}^m : \sum_{k=1}^m \beta_k g_k(e_i) = 0\Big\}.$$

Assume that $H_1, H_2, \ldots H_r$ are distinct hyperplanes, where $r \leq n$. For each $i = 1, 2 \ldots r$, consider the sets H_i^+ and H_i^- given by

$$H_i^+ = \Big\{(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{R}^m : \sum_{k=1}^m \beta_k g_k(e_i) > 0\Big\}$$

$$H_i^- = \Big\{(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{R}^m : \sum_{k=1}^m \beta_k g_k(e_i) < 0\Big\}.$$

Observe that $H_i^+ \cap H_i^- = \emptyset$ and $H_i^+ \cup H_i \cup H_i^- = \mathbb{R}^m$, for each $i = 1, 2, \ldots r$. Consider the set $K_j = H_1^{\delta_{j1}} \cap H_2^{\delta_{j2}} \cap \ldots \cap H_r^{\delta_{jr}}$, where $\delta_{ji} \in \{+, -\}$ for each $i = 1, 2, \ldots r$. It is immediate that there are atmost 2^r number of such sets. We assume that $\pm K_1, \pm K_2, \ldots, \pm K_q$ are the nonempty such sets. Then

$$\mathbb{R}^m = \Big(\cup_{i=1}^{q} (K_i \cup -K_i) \Big) \bigcup \Big(\cup_{j=1}^{r} H_j \Big) = K \cup H,$$

where $K = \cup_{i=1}^{q}(K_i \cup -K_i)$ and $H = \cup_{j=1}^{r} H_j$.

Step 2 : We find the norm attaining set of functionals of the form $\sum_{k=1}^{m} \beta_k g_k$, where $(\beta_1, \beta_2, \ldots, \beta_n) \in K$. We first associate each of the nonempty sets $K_s (1 \leq s \leq q)$ with an extreme point of $B_{\ell_\infty^n}$. For any $\widetilde{\beta} = (\beta_1, \beta_2, \ldots, \beta_m) \in K_s$, let us construct $\widetilde{x_s} = (x_1, x_2, \ldots, x_n) \in S_{\ell_\infty^n}$ where

$$\begin{aligned} x_t &= 1, \quad \widetilde{\beta} \in H_t^+ \\ &= -1, \quad \widetilde{\beta} \in H_t^-. \end{aligned}$$

Clearly, $\widetilde{x_s} \in Ext(B_{\ell_\infty^n})$. Note that construction of $\widetilde{x_s}$ is independent of $\widetilde{\beta}$, for if $\widetilde{\beta} \in (H_{i_1}^+ \cap H_{i_2}^+ \ldots \cap H_{i_s}^+) \cap (H_{j_1}^- \cap H_{j_2}^- \ldots \cap H_{j_t}^-)$, where $1 \leq s+t \leq r$, then for any $\widetilde{\omega} \in K_s$, we have $\widetilde{\omega} \in (H_{i_1}^+ \cap H_{i_2}^+ \ldots \cap H_{i_s}^+) \cap (H_{j_1}^- \cap H_{j_2}^- \ldots \cap H_{j_t}^-)$. Thus with each K_i we associate an extreme point $\widetilde{x_i}$, where $1 \leq i \leq q$. Let $\mathcal{N} = \{\pm\widetilde{x_1}, \pm\widetilde{x_2}, \ldots, \pm\widetilde{x_q}\}$. We show that $\mathcal{N} = \cup_{(\beta_1, \beta_2, \ldots, \beta_m) \in K} M_{\sum_{k=1}^{m} \beta_k g_k}$. Let $(\beta_1, \beta_2, \ldots, \beta_m) \in K$, then $(\beta_1, \beta_2, \ldots, \beta_m) \in K_s$, for some s. Consider $g = \sum_{k=1}^{m} \beta_k g_k$. Then $g(\widetilde{x_s}) = \sum_{k=1}^{m} \beta_k$
$g_k(e_i) x_i > 0$. We show that $M_g = \{\pm\widetilde{x_s}\}$. Let $\widetilde{y} = (y_1, y_2, \ldots, y_n) \in M_g \cap Ext(B_{\ell_\infty^n})$.
Therefore,

$$|g(\widetilde{y})| = \Big| \sum_{k=1}^{m} \beta_k g_k(\widetilde{y}) \Big| = \Big| \Big(\sum_{k=1}^{m} \beta_k g_k(e_1) \Big) \Big| + \Big| \Big(\sum_{k=1}^{m} \beta_k g_k(e_2) \Big) \Big| + \ldots + \Big| \Big(\sum_{k=1}^{m} \beta_k g_k(e_n) \Big) \Big|$$

which implies

$$\begin{aligned} &\Big| \Big(\sum_{k=1}^{m} \beta_k g_k(e_1) \Big) y_1 + \Big(\sum_{k=1}^{m} \beta_k g_k(e_2) \Big) y_2 + \ldots + \Big(\sum_{k=1}^{m} \beta_k g_k(e_n) \Big) y_n \Big| \\ &= \Big| \Big(\sum_{k=1}^{m} \beta_k g_k(e_1) \Big) \Big| + \Big| \Big(\sum_{k=1}^{m} \beta_k g_k(e_2) \Big) \Big| + \ldots + \Big| \Big(\sum_{k=1}^{m} \beta_k g_k(e_n) \Big) \Big|. \end{aligned}$$

The last equality is satisfied if and only if each $\Big(\sum_{k=1}^{m} \beta_k g_k(e_i) \Big) y_i$ have the same sign, which in turn is satisfied if and only if $\widetilde{y} = \pm\widetilde{x_s}$. Thus $M_g = \{\pm\widetilde{x_s}\}$. Therefore, whenever $(\alpha_1, \alpha_2, \ldots, \alpha_m) \in K_i$, for some $1 \leq i \leq q$, we have $M_{\sum_{k=1}^{m} \alpha_k g_k} = \{\pm\widetilde{x_i}\}$. Thus $\mathcal{N} = \cup_{(\beta_1, \beta_2, \ldots, \beta_m) \in K} M_{\sum_{k=1}^{m} \beta_k g_k}$.

Step 3 : We deal with functionals of the form $\sum_{k=1}^m \gamma_k g_k$, where $\widetilde{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in H$. Let us assume that

$$\widetilde{\gamma} \in H_{b_1} \cap H_{b_2} \cap \ldots \cap H_{b_u} \cap H_{c_1}^+ \cap H_{c_2}^+ \cap \ldots \cap H_{c_s}^+ \cap H_{d_1}^- \cap H_{d_2}^- \ldots \cap H_{d_t}^-,$$

where $u + s + t = r$.

Let us now consider the set $D_{\widetilde{\gamma}} = (\cap_{i=1}^s H_{c_i}^+) \cap (\cap_{i=1}^t H_{d_i}^-)$. Now it is easy to observe that $\widetilde{\gamma} \in D_{\widetilde{\gamma}}$ and $D_{\widetilde{\gamma}}$ is an open set of $\mathbb{R}^m$. Take

$$\widetilde{\eta} = (\eta_1, \eta_2, \ldots, \eta_n) \in D_{\widetilde{\gamma}} \setminus (\cup_{i=1}^u H_{b_i}).$$

Then $\sum_{k=1}^m \eta_k g_k(e_l) > 0$, for any $l \in \{c_1, c_2, \ldots, c_s\}$ and $\sum_{k=1}^m \eta_k g_k(e_l) < 0$, for any $l \in \{d_1, d_2, \ldots, d_t\}$. It is easy to observe that $\widetilde{\eta} \in K_p$, for some $1 \leq p \leq q$. Therefore, $M_{\sum_{k=1}^m \eta_k g_k} = \{\pm \widetilde{x_p} = (x_1, x_2, \ldots, x_n)\}$ as claimed before. Observe that $x_l = 1$, for any $l \in \{c_1, c_2, \ldots, c_s\}$ and $x_l = -1$, for all $l \in \{d_1, d_2, \ldots, d_t\}$. By a straightforward calculation it is easy to observe that $\pm \widetilde{x_p} \in M_{\sum_{k=1}^m \gamma_k g_k}$.

Step 4 : We show that $\mathcal{N}$ is the unique minimal norming set of $\mathbb{Y}$. From the previous two steps it follows that $\mathcal{N}$ is a norming set of $\mathbb{Y}$. Let us consider a symmetric set $\mathcal{M} \subsetneq \mathcal{N}$ and also assume that $\pm \widetilde{x_j} \in \mathcal{N} \setminus \mathcal{M}$. It can be clearly seen that whenever $\widetilde{\beta} \in K_j$, $\mathcal{M} \cap M_{\sum_{k=1}^m \beta_k g_k} = \emptyset$. This implies that $\mathcal{N}$ is a minimal norming set of $\mathbb{Y}$. From Step 2 it follows that $\mathcal{N}$ is the unique minimal norming set of $\mathbb{Y}$.

□

We next explore the converse of the previous result.

Theorem 3.3. *Let $\mathbb{Y}$ be a subspace of ℓ_1^n and let $\mathcal{A} = \{\widetilde{a_1}, \widetilde{a_2}, \ldots, \widetilde{a_m}\}$ be a basis of $\mathbb{Y}$. If the minimal norming set of $\mathbb{Y}$ is unique then $\mathcal{Z}_{\mathcal{A}} = \emptyset$.*

Proof. Let us assume that the minimal norming set $\mathcal{N}$ of $\mathbb{Y}$ is unique. Suppose on the contrary that $j \in \mathcal{Z}_{\mathcal{A}}$, for some $1 \leq j \leq n$. This implies that $g(e_j) = 0$, for any $g \in \psi(\mathbb{Y})$. Suppose that $\mathcal{N} = \{\pm \widetilde{x_1}, \pm \widetilde{x_2}, \ldots, \pm \widetilde{x_q}\}$ is a minimal norming set of $\mathbb{Y}$, where $\widetilde{x_k} = (x_1^k, x_2^k, \ldots, x_n^k)$, for $1 \leq k \leq q$. Let us now consider

$$\widetilde{y_1} = (x_1^1, x_2^1, \ldots, x_{j-1}^1, -x_j^1, x_{j+1}^1, \ldots, x_n^1) \in Ext(B_{\ell_\infty^n}).$$

It can be easily observed that for any $g \in \psi(\mathbb{Y})$, $\widetilde{y_1} \in M_g$ if and only if $\widetilde{x_1} \in M_g$. Therefore, $\mathcal{N}_1 = \{\pm \widetilde{y_1}, \pm \widetilde{x_2}, \ldots, \pm \widetilde{x_q}\}$ is a norming set of $\mathbb{Y}$. Since $\widetilde{y_1} \notin \{\pm \widetilde{x_2}, \ldots, \pm \widetilde{x_q}\}$, $\mathcal{N}_1 (\neq \mathcal{N})$ is a minimal norming set of $\mathbb{Y}$. This contradicts the assumption that the minimal norming set of $\mathbb{Y}$ is unique. □

3.4 Best coapproximation in ℓ_1^n

Now we are in a position to present the characterization of the best coapproximation in $\mathbb{Y}$. This is given in terms of a system of linear equations which clearly illustrates its computational effectiveness.

Theorem 3.4. *Let $\mathbb{Y}$ be a subspace of ℓ_1^n and let $\mathcal{A} = \{\widetilde{a_1}, \widetilde{a_2}, \ldots, \widetilde{a_m}\}$ be a basis of $\mathbb{Y}$ with $\mathcal{Z}_{\mathcal{A}} = \emptyset$. Suppose that $\mathcal{N} = \{\pm\widetilde{x_1}, \pm\widetilde{x_2}, \ldots, \pm\widetilde{x_q}\}$ is the minimal norming set of $\mathbb{Y}$, where $\widetilde{x_k} = (x_1^k, x_2^k, \ldots, x_n^k)$, for any $1 \leq k \leq q$. Then given $\widetilde{b} = (b_1, b_2, \ldots, b_n) \in \ell_1^n$, $\sum_{k=1}^m \alpha_k \widetilde{a_k}$ is a best coapproximation to $\widetilde{b}$ out of $\mathbb{Y}$ if and only if $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ satisfy the following relations:*

$$\alpha_1 \sum_{i=1}^n a_i^1 x_i^p + \alpha_2 \sum_{i=1}^n a_i^2 x_i^p + \ldots + \alpha_m \sum_{i=1}^n a_i^m x_i^p = \sum_{i=1}^n b_i x_i^p,$$

for any $p \in \{1, 2, \ldots, q\}$.

Proof. Suppose that $\psi(\widetilde{a_i}) = g_i$, for any $1 \leq i \leq m$ and $\psi(b) = f$. We observe that $\sum_{k=1}^m \alpha_k \widetilde{a_k}$ being a best coapproximation to $\widetilde{b}$ out of $\mathbb{Y}$ is equivalent to $\sum_{k=1}^m \alpha_k g_k$ being a best coapproximation to f out of $\psi(\mathbb{Y})$.

Let us first prove the necessary part of the theorem. Since $\{\pm\widetilde{x_1}, \pm\widetilde{x_2}, \ldots, \pm\widetilde{x_q}\}$ is the minimal norming set of $\mathbb{Y}$, it can be easily observed that for any $\widetilde{x_s}$, there exists $\widetilde{\beta} = (\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{R}^m$ such that $M_{\sum_{k=1}^m \beta_k g_k} = \{\pm\widetilde{x_s}\}$. It follows from Theorem 3.1 that

$$M_{\sum_{k=1}^m \beta_k g_k} \cap \ker(f - \sum_{k=1}^m \alpha_k g_k) \neq \emptyset.$$

Therefore, $\widetilde{x_s} \in \ker(f - \sum_{k=1}^m \alpha_k g_k)$, i.e., $(f - \sum_{k=1}^m \alpha_k g_k)\widetilde{x_s} = 0$, which implies

$$f(\widetilde{x_s}) = \alpha_1 g_1(\widetilde{x_s}) + \alpha_2 g_2(\widetilde{x_s}) + \ldots + \alpha_m g_m(\widetilde{x_s}).$$

This is equivalent to

$$\alpha_1 \sum_{i=1}^n g_1(e_i) x_i^s + \alpha_2 \sum_{i=1}^n g_2(e_i) x_i^s + \ldots + \alpha_m \sum_{i=1}^n g_m(e_i) x_i^s = \sum_{i=1}^n f(e_i) x_i^s.$$

Similarly, we can observe that for all $p \in \{1, 2, \ldots, q\}$,

$$\alpha_1 \sum_{i=1}^{n} g_1(e_i)x_i^p + \alpha_2 \sum_{i=1}^{n} g_2(e_i)x_i^p + \ldots + \alpha_m \sum_{i=1}^{n} g_m(e_i)x_i^p = \sum_{i=1}^{n} f(e_i)x_i^p,$$

which implies,

$$\alpha_1 \sum_{i=1}^{n} a_i^1 x_i^p + \alpha_2 \sum_{i=1}^{n} a_i^2 x_i^p + \ldots + \alpha_m \sum_{i=1}^{n} a_i^m x_i^p = \sum_{i=1}^{n} b_i x_i^p,$$

for any $p \in \{1, 2, \ldots, q\}$. This completes the necessary part of the theorem.

We now prove the sufficient part of the theorem. From the hypothesis, we obtain that $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ satisfy the following relations:

$$\alpha_1 \sum_{i=1}^{n} g_1(e_i)x_i^t + \alpha_2 \sum_{i=1}^{n} g_2(e_i)x_i^t + \ldots + \alpha_1 \sum_{i=1}^{n} g_m(e_i)x_i^t = \sum_{i=1}^{n} f(e_i)x_i^t, \tag{3.2}$$

for any $t \in \{1, 2, \ldots, q\}$. Now

$$(f - \sum_{k=1}^{m} \alpha_k g_k)\widetilde{x_t} = f(\widetilde{x_t}) - \Big\{\alpha_1 g_1(\widetilde{x_t}) + \alpha_2 g_2(\widetilde{x_t}) + \ldots + \alpha_m g_m(\widetilde{x_t})\Big\}.$$

Therefore, using equation (3.2), it is immediate that $\widetilde{x_t} \in \ker(f - \sum_{k=1}^{m} \alpha_k g_k)$, for any $t \in \{1, 2, \ldots, q\}$. For any $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$, not all zero, there exists $s \in \{1, 2, \ldots, q\}$ such that $\widetilde{x_s} \in M_{\sum_{k=1}^{m} \beta_k g_k}$. Therefore,

$$\widetilde{x_s} \in \ker(f - \sum_{k=1}^{m} \alpha_k g_k) \cap M_{\sum_{k=1}^{m} \beta_k g_k}.$$

Therefore, from Theorem 3.1, the sufficient part of the theorem follows directly.

□

Combining Theorem 3.4 with the theoretical characterization of best coapproximation in terms of norm-1 projections, as given in [26, 33], we get the following result.

Corollary 3.1. *Let $\mathbb{Y}$ be a subspace of ℓ_1^n and let $\mathcal{A} = \{\widetilde{a_1}, \widetilde{a_2}, \ldots, \widetilde{a_m}\}$ be a basis of $\mathbb{Y}$ with $\mathcal{Z}_{\mathcal{A}} = \emptyset$. Suppose that $\mathcal{N} = \{\pm\widetilde{x_1}, \pm\widetilde{x_2}, \ldots, \pm\widetilde{x_q}\}$ is the minimal norming set of $\mathbb{Y}$, where $\widetilde{x_k} = (x_1^k, x_2^k, \ldots, x_n^k)$, for any $1 \leq k \leq q$. Then given $\widetilde{b} = (b_1, b_2, \ldots, b_n) \in \ell_1^n$, there exists a norm-1 projection from span$\{\widetilde{b}, \mathbb{Y}\}$ to $\mathbb{Y}$ if and only if there exist $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ satisfy the* *following relations:*

$$\alpha_1 \sum_{i=1}^{n} a_i^1 x_i^p + \alpha_2 \sum_{i=1}^{n} a_i^2 x_i^p + \ldots + \alpha_m \sum_{i=1}^{n} a_i^m x_i^p = \sum_{i=1}^{n} b_i x_i^p,$$

for any $p \in \{1, 2, \ldots, q\}$. *Moreover, if* $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ *satisfy the above system of linear equations then* $P(\widetilde{a} + \gamma\widetilde{b}) = \widetilde{a} + \gamma(\sum_{i=1}^{m} \alpha_i \widetilde{a_i})$ *is the norm 1 projection, for any* $\widetilde{a} \in \mathbb{Y}$.

We now present an explicit numerical example to illustrate the applicability of Theorem 3.4 towards solving the best coapproximation problem in ℓ_1^n, from a computational point of view.

Example 3.5. *Find the best coapproximation(s) to any given* $\widetilde{b} \in \ell_1^6$ *out of the subspace* $\mathbb{Y} = span\,\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}\}$ *of* ℓ_1^6, *where* $\widetilde{a_1} = (\ 4,\ 2,\ 1,\ -1,\ -4,\ 4)$, $\widetilde{a_2} = (\ -1,\ 3,\ 5,\ 2,\ 1,\ 6)$, $\widetilde{a_3} = (\ 1,\ 4,\ 2,\ 1,\ -1,\ 8) \in \ell_1^6$.

Step 1: *Let* $\psi(\widetilde{a_i}) = g_i \in (\ell_\infty^n)^*$, *where* ψ *is the canonical isometric isomorphism from* ℓ_1^6 *to* $(\ell_\infty^6)^*$. *Here, for any* $(x_1, x_2, \ldots, x_6) \in \ell_\infty^6$,

$$\begin{aligned} g_1(x_1, x_2, x_3, x_4, x_6) &= 4x_1 + 2x_2 + x_3 - x_4 - 4x_5 + 4x_6, \\ g_2(x_1, x_2, x_3, x_4, x_6) &= -x_1 + 3x_2 + 5x_3 + 2x_4 + x_5 + 6x_6, \\ g_3(x_1, x_2, x_3, x_4, x_6) &= x_1 + 4x_2 + 2x_3 + x_4 - x_5 + 8x_6. \end{aligned}$$

We first observe that $\mathcal{Z}_{\mathcal{A}} = \emptyset$. *From Theorem 3.2, suppose that* $\mathcal{N}$ *is the unique minimal norming set of* $\mathbb{Y}$.

Step 2: *We observe that the* 1*-st,* 2*-nd,* 3*-rd and* 4*-th positions can be taken as the nonequivalent components. The hyperplanes corresponding to each components are:*

$$\begin{aligned} H_1 &= \Big\{(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3 : 4\beta_1 - \beta_2 + \beta_3 = 0\Big\}, \\ H_2 &= \Big\{(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3 : 2\beta_1 + 3\beta_2 + 4\beta_3 = 0\Big\}, \\ H_3 &= \Big\{(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3 : \beta_1 + 5\beta_2 + 2\beta_3 = 0\Big\}, \\ H_4 &= \Big\{(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3 : -\beta_1 + 2\beta_2 + \beta_3 = 0\Big\}, \\ H_5 &= \Big\{(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3 : -4\beta_1 + \beta_2 - \beta_3 = 0\Big\}, \\ H_6 &= \Big\{(\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3 : 4\beta_1 + 6\beta_2 + 8\beta_3 = 0\Big\}. \end{aligned}$$

Clearly, $H_5 = H_1, H_6 = H_2$ *and* $H_5^+ = H_1^-$, $H_5^- = H_1^+$; $H_6^+ = H_2^+, H_6^- = H_2^-$.

Step 3: *To solve the best coapproximation problem with the help of Theorem 3.4, we first need to find a basis of span* $\mathcal{N}$*. We observe that there are four nonequivalent positions, and therefore, from Proposition 3.3 we note that* $\dim(span\ \mathcal{N}) \leq 4$.

As mentioned in Theorem 3.2, we consider the sets $K_i = H_1^{\delta_{i_1}} \cap H_2^{\delta_{i_2}} \cap H_3^{\delta_{i_3}} \cap H_4^{\delta_{i_4}}$*, where* $\delta_{i_j} \in \{+,-\}$*, for any* $j \in \{1,2,3,4\}$*. Although there are* 2^4 *number of possible* K_i*'s, it is evident that we only need to take account of the nonempty* K_i*'s. Moreover, we associate each of these nonempty* K_i*'s with an extreme point* $\widetilde{x_i}$ *of* $B_{\ell_\infty^6}$*, as mentioned in Theorem 3.2.*

Suppose that $K_1 = H_1^+ \cap H_2^+ \cap H_3^+ \cap H_4^+$ *and it is straightforward to verify that* $(1,2,3) \in K_1$*. Therefore, we obtain* $\widetilde{x_1} = (1,1,1,1,-1,1) \in Ext(B_{\ell_\infty^6})$.

In a similar manner, we take

$$K_2 = H_1^+ \cap H_2^+ \cap H_3^+ \cap H_4^-, \quad (4,-1,1) \in K_2.$$

Therefore, $\widetilde{x_2} = (1,1,1,-1,-1,1)$*. Again*

$$K_3 = H_1^+ \cap H_2^+ \cap H_3^- \cap H_4^-, \quad (0,-1,\frac{3}{2}) \in K_3.$$

So, $\widetilde{x_3} = (1,1,-1,-1,-1,1)$*. Also take*

$$K_4 = H_1^+ \cap H_2^- \cap H_3^- \cap H_4^-, \quad (1,0,-1) \in K_4.$$

Therefore, we have $\widetilde{x_4} = (1,-1,-1,-1,-1,-1)$.

From Theorem 3.2, it is now immediate that $\widetilde{x_1}, \widetilde{x_2}, \widetilde{x_3}, \widetilde{x_4} \in \mathcal{N}$*. It is straightforward to check that* $\{\widetilde{x_1}, \widetilde{x_2}, \widetilde{x_3}, \widetilde{x_4}\}$ *is linearly independent. Therefore,* $\{\pm(1,1,1,1,-1,1), \pm(1,1,1,-1,-1,1), \pm(1,1,-1,-1,-1,1), \pm(1,-1,-1,-1,-1,-1)\}$ *is a basis of span* $\mathcal{N}$*.*

Step 4: *In this final step, by considering a given* $\widetilde{b} \in \ell_1^6$ *and thereafter applying Theorem 3.4, we obtain the best coapproximation to* $\widetilde{b}$ *out of* $\mathbb{Y}$*. In order to illustrate the various possibilities arising in the best coapproximation problem in* ℓ_1^6*, it suffices to consider the following two particular cases.*

Case 1 : Let $\widetilde{b_1} = (1,2,3,4,5,6) \in \ell_1^6$*. Then from Theorem 3.4,* $\sum_{i=1}^{3} \alpha_i g_i$ *is a best coapproximation to* $\widetilde{b}_1$ *out of* $\mathbb{Y}$ *if and only if* $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ *satisfies the following relations:*

$$\begin{aligned} 14\alpha_1 + 14\alpha_2 + 17\alpha_3 &= 11 \\ 16\alpha_1 + 10\alpha_2 + 15\alpha_3 &= 3 \end{aligned}$$

$$\begin{aligned} 14\alpha_1 + 11\alpha_3 &= -3 \\ 2\alpha_1 - 18\alpha_2 - 13\alpha_3 &= -19. \end{aligned}$$

Since there exist no such $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ satisfying the above relations, it follows that

$$\mathcal{R}_{\mathbb{Y}}(\widetilde{b_1}) = \emptyset.$$

Case 2 : *Let $\widetilde{b_2} = (5, 4, 0, 0, 1, 5) \in \ell_1^6$. Then from Theorem 3.4, $\sum_{i=1}^{3} \alpha_i \widetilde{a_i}$ is a best coapproximation to $\widetilde{b_2}$ out of $\mathbb{Y}$ if and only if $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ satisfies the following relations:*

$$\begin{aligned} 14\alpha_1 + 14\alpha_2 + 17\alpha_3 &= 13 \\ 16\alpha_1 + 10\alpha_2 + 15\alpha_3 &= 13 \\ 14\alpha_1 \qquad\quad + 11\alpha_3 &= 13 \\ 2\alpha_1 - 18\alpha_2 - 13\alpha_3 &= -5. \end{aligned}$$

Since there exist unique $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ satisfying the above relations, the best coapproximation to $\widetilde{b_2}$ out of $\mathbb{Y}$ is unique. Moreover, $\alpha_1 = \frac{1}{7}, \alpha_2 = -\frac{3}{7}, \alpha_3 = 1$ and therefore

$$\mathcal{R}_{\mathbb{Y}}(\widetilde{b_2}) = \frac{1}{7}\widetilde{a_1} - \frac{3}{7}\widetilde{a_2} + \widetilde{a_3} = (2, 3, 0, 0, -2, 6).$$

We next observe with the following remark.

Remark 3.6. *It is already known from [26, 33] that whenever the best coapproximation exists from a element $\widetilde{b}$ to a subspace $\mathbb{Y}$ of ℓ_1^n, then there exist a norm-1 projection from span$\{\widetilde{b}, \mathbb{Y}\}$ to $\mathbb{Y}$. However, it is also natural to look for the explicit description of the concerned projection. In view of the method developed here, we can now find the projection map explicitly. If we consider the subspace $\mathbb{Y}$ and element $\widetilde{b_2}$ as in Example 3.5, then the norm-1 projection P from span$\{\widetilde{b_2}, \mathbb{Y}\}$ to $\mathbb{Y}$ is given by:*

$$P(\widetilde{b_2}) = P(5, 4, 0, 0, 1, 5) = (2, 3, 0, 0, -2, 6); \qquad P(y) = y, \forall y \in \mathbb{Y}.$$

To complete this study we now investigate the subspaces where the zero set is non-empty. Our idea is to truncate the subspace into a subspace where the zero set is empty. This process is formalized through Definition 3.2. In the next theorem we provide a simple sufficient condition of the best coapproximation problem for the subspaces where the zero set is non-empty.

Theorem 3.7. *Let $\mathbb{Y}$ be a subspace of ℓ_1^n and let $\mathcal{A} = \{\widetilde{a_1}, \widetilde{a_2}, \ldots, \widetilde{a_m}\}$ be a basis of $\mathbb{Y}$ with $|\mathcal{Z}_{\mathcal{A}}| = r > 0$. Then for any $\widetilde{b} \in \ell_1^n$, $\sum_{i=1}^{m} \alpha_i \widetilde{a_i}$ is a best coapproximation to $\widetilde{b}$ out of $\mathbb{Y}$ if $\sum_{i=1}^{m} \alpha_i \sigma(\widetilde{a_i})$*

is a best coapproximation to $\sigma(\widetilde{b})$ *out of* $\sigma(\mathbb{Y})$*, where* $\sigma(\mathbb{Y}) = span\{\sigma(\widetilde{a_1}), \sigma(\widetilde{a_2}), \ldots, \sigma(\widetilde{a_m})\}$.

Proof. Since $\sum_{i=1}^{m} \alpha_i \sigma(\widetilde{a_i})$ is a best coapproximation to $\sigma(\widetilde{b})$ out of $\sigma(\mathbb{Y})$, so for any $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$, we have

$$\begin{aligned}\|\sum_{i=1}^{m} \beta_i \widetilde{a_i} - \sum_{i=1}^{m} \alpha_i \widetilde{a_i}\| = \|\sum_{i=1}^{m} \beta_i \sigma(\widetilde{a_i}) - \sum_{i=1}^{m} \alpha_i \sigma(\widetilde{a_i})\| &\leq \|\sum_{i=1}^{m} \beta_i \sigma(\widetilde{a_i}) - \sigma(\widetilde{b})\| \\ &= \|\sigma(\sum_{i=1}^{m} \beta_i \widetilde{a_i} - b)\| \\ &\leq \|\sum_{i=1}^{m} \beta_i \widetilde{a_i} - \widetilde{b}\|.\end{aligned}$$

In other words, $\sum_{i=1}^{m} \alpha_i \widetilde{a_i}$ is a best coapproximation to $\widetilde{b}$ out of $\mathbb{Y}$. This establishes our theorem. □

Remark 3.8. *Observe that the zero set corresponding to a basis of* $\sigma(\mathbb{Y})$ *is empty and so following the method discussed in Theorem 3.4, we can find the best coaproximation to* $\sigma(\widetilde{b})$ *out of* $\sigma(\mathbb{Y})$*, which in turn allows us to exactly find the best coapproximation to* $\widetilde{b}$ *out of* $\mathbb{Y}$.

In order to characterize the best coapproximation problem in ℓ_1^n, we require the following lemma.

Lemma 3.1. *Let* $\mathbb{Y}$ *be a subspace of* ℓ_1^n *and let* $\mathcal{A} = \{\widetilde{a_1}, \widetilde{a_2}, \ldots, \widetilde{a_m}\}$ *be a basis of* $\mathbb{Y}$ *with* $|\mathcal{Z}_{\mathcal{A}}| = r > 0$. *Suppose that for* $\widetilde{b} \in \ell_1^n$*, there exists no best coapproximation to* $\sigma(\widetilde{b})$ *out of* $\sigma(\mathbb{Y}) = span\{\sigma(\widetilde{a_1}), \sigma(\widetilde{a_2}), \ldots, \sigma(\widetilde{a_m})\}$. *Then there exists* $\delta > 0$ *such that for any* $y \in \mathcal{B}_\delta(\rho(\widetilde{b})) \bigcap \mathcal{P}_{\widetilde{b}}$*, there exists no best coapproximation to* y *out of* $\mathbb{Y}$.

Proof. Since there exists no best coapproximation to $\sigma(\widetilde{b})$ out of $\sigma(\mathbb{Y})$, then for any $(\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m$, there exists $(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{R}^m$ such that

$$\|\sum_{i=1}^{m} \beta_i \sigma(\widetilde{a_i}) - \sum_{i=1}^{m} \alpha_i \sigma(\widetilde{a_i})\| \ > \ \|\sum_{i=1}^{m} \beta_i \sigma(\widetilde{a_i}) - \sigma(\widetilde{b})\| = \|\sigma(\sum_{i=1}^{m} \beta_i \widetilde{a_i} - \widetilde{b})\|.$$

It is easy to observe that for any $\widetilde{u} \in \ell_1^n$, if $\rho(\widetilde{u}) = \widetilde{u}$ then $\|\sigma(\widetilde{u})\| = \|\widetilde{u}\|$. Since $\rho\big(\sum_{i=1}^{m} \beta_i \widetilde{a_i} - \rho(\widetilde{b})\big) = \sum_{i=1}^{m} \beta_i \widetilde{a_i} - \rho(\widetilde{b})$, we have

$$\|\sigma\left(\sum_{i=1}^{m} \beta_i \widetilde{a_i} - \widetilde{b}\right)\| = \|\sigma\left(\sum_{i=1}^{m} \beta_i \widetilde{a_i} - \rho(\widetilde{b})\right)\| = \|\sum_{i=1}^{m} \beta_i \widetilde{a_i} - \rho(\widetilde{b})\|$$

and therefore,

$$\|\sum_{i=1}^{m} \beta_i \widetilde{a_i} - \sum_{i=1}^{m} \alpha_i \widetilde{a_1}\| = \|\sum_{i=1}^{m} \beta_i \sigma(\widetilde{a_i}) - \sum_{i=1}^{m} \alpha_i \sigma(\widetilde{a_i})\| > \|\sum_{i=1}^{m} \beta_i \widetilde{a_i} - \rho(\widetilde{b})\|.$$

In other words, there exists no best coapproximation to $\rho(\widetilde{b})$ out of $\mathbb{Y}$. Since for any $\widetilde{u} \in \ell_1^n$, $\mathcal{R}_{\mathbb{Y}}(\widetilde{u})$ is a compact set [31], it is immediate that $\mathcal{D}(\mathcal{R}_{\mathbb{Y}})$ is closed. Therefore, there exists an open ball of radius $\delta > 0$ (say) centered at $\rho(\widetilde{b})$, $\mathcal{B}_\delta(\rho(\widetilde{b}))$ such that for any $y \in \mathcal{B}_\delta(\rho(\widetilde{b})) \bigcap \mathcal{P}_{\widetilde{b}}$, there exists no best coapproximation to y out of $\mathbb{Y}$. This proves our lemma.

□

We next characterize the existence of the best coapproximation(s) in ℓ_1^n in the following theorem.

Theorem 3.9. *Let $\mathbb{Y}$ be a subspace of ℓ_1^n and let $\mathcal{A} = \{\widetilde{a_1}, \widetilde{a_2}, \ldots, \widetilde{a_m}\}$ be a basis of $\mathbb{Y}$ with $|\mathcal{Z}_{\mathcal{A}}| = r > 0$. Suppose that for $\widetilde{b} \in \ell_1^n$, there exists no best coapproximation to $\sigma(\widetilde{b})$ out of $\sigma(\mathbb{Y}) = span\{\sigma(\widetilde{a_1}), \sigma(\widetilde{a_2}), \ldots, \sigma(\widetilde{a_m})\}$. Then there exists $\delta_0(> 0)$ satisfying the following:*

(i) for any $y \in \mathcal{P}_{\widetilde{b}}$ such that $\|y - \rho(\widetilde{b})\| < \delta_0$, there exists no best coapproximation to y out of $\mathbb{Y}$,

(ii) for any $y \in \mathcal{P}_{\widetilde{b}}$ such that $\|y - \rho(\widetilde{b})\| \geq \delta_0$, there exists a best coapproximation to y out of $\mathbb{Y}$.

Proof. Let us define the set

$$S := \left\{\delta \in \mathbb{R} : \mathcal{R}_{\mathbb{Y}}(y) = \emptyset \ \forall \ y \in \mathcal{B}_\delta(\rho(\widetilde{b})) \bigcap \mathcal{P}_{\widetilde{b}}\right\}.$$

To prove the theorem, we only need to show that S has a upper bound. From Lemma 3.1, it is assured that $S \neq \emptyset$ and if there exists no best coapproximation to $\sigma(\widetilde{b})$ out of $\sigma(\mathbb{Y})$ then there exists no best coapproximation to $\rho(\widetilde{b})$ out of $\mathbb{Y}$. It is easy to observe that for all $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$,

$$\|\sum_{i=1}^{m} \beta_i \widetilde{a_i}\| \leq \|\sum_{i=1}^{m} \beta_i \widetilde{a_i} - \rho(\widetilde{b})\| + \|\rho(\widetilde{b})\|.$$

Suppose that $\widetilde{b} = (b_1, b_2, \ldots, b_n)$. For any $y \in \mathcal{P}_{\widetilde{b}}$, we observe that $\rho(y) = \rho(\widetilde{b}) \in \ell_1^n$, i.e., $y_i = b_i$, for any $i \notin \mathcal{Z}_{\mathcal{A}}$. We now choose $y \in \mathcal{P}_{\widetilde{b}}$ such that $\|y - \rho(\widetilde{b})\| = \|\rho(\widetilde{b})\|$ then

$$\|\sum_{i=1}^{m} \beta_i \widetilde{a_i}\| \leq \|\sum_{i=1}^{m} \beta_i \widetilde{a_i} - \rho(\widetilde{b})\| + \|\rho(\widetilde{b})\| = \|\sum_{i=1}^{m} \beta_i \widetilde{a_i} - \rho(\widetilde{b})\| + \|y - \rho(\widetilde{b})\| \tag{3.3}$$

for all $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$. Now we observe that

$$\|\sum_{i=1}^{m} \beta_i \widetilde{a_i} - y\| \quad = \quad \sum_{j=1}^{n} \left| \left(\sum_{i=1}^{m} \beta_i a_j^i - y_j \right) \right|$$

$$= \sum_{j \notin \mathcal{Z}_{\mathcal{A}}} \left| \left(\sum_{i=1}^{m} \beta_i a_j^i - y_j \right) \right| + \sum_{j \in \mathcal{Z}_{\mathcal{A}}} \left| \left(\sum_{i=1}^{m} \beta_i a_j^i - y_j \right) \right|$$
$$= \sum_{j \notin \mathcal{Z}_{\mathcal{A}}} \left| \left(\sum_{i=1}^{m} \beta_i a_j^i - b_j \right) \right| + \sum_{j \in \mathcal{Z}_{\mathcal{A}}} |y_j|$$

Therefore, we can easily obtain that

$$\| \sum_{i=1}^{m} \beta_i \widetilde{a_i} - y \| = \| \sum_{i=1}^{m} \beta_i \widetilde{a_i} - \rho(\widetilde{b}) \| + \| y - \rho(\widetilde{b}) \|. \tag{3.4}$$

From equations (3.3) and (3.4) we have

$$\| \sum_{i=1}^{m} \beta_i \widetilde{a_i} \| \leq \| \sum_{i=1}^{m} \beta_i \widetilde{a_i} - y \|.$$

In other words, $\theta = (0, 0, \ldots, 0) \in \ell_1^n$ is a best coapproximation to y out of $\mathbb{Y}$. Take $\gamma > \|\rho(\widetilde{b})\|$. Therefore, $y \in \mathcal{B}_\gamma(\rho(\widetilde{b})) \bigcap \mathcal{P}_{\widetilde{b}}$ and consequently γ is an upper bound of S. Let $\sup S = \delta_0$. Hence the theorem. $\square$

3.4.1 Coproximinal and co-Chebyshev subspaces in ℓ_1^n

We next obtain an immediate corollary from Theorem 3.4, which guarantees the uniqueness of best coapproximation to a element in ℓ_1^n out of a subspace of ℓ_1^n, provided it exists.

Theorem 3.10. *Let $\mathbb{Y}$ be a subspace of ℓ_1^n and let $\mathcal{A} = \{\widetilde{a_1}, \widetilde{a_2}, \ldots, \widetilde{a_m}\}$ be a basis of $\mathbb{Y}$ with $\mathcal{Z}_{\mathcal{A}} = \emptyset$. For any given $\widetilde{b} \in \ell_1^n$, if there exists a best coapproximation to $\widetilde{b}$ out of $\mathbb{Y}$ then it is unique.*

Proof. Let $\{\pm\widetilde{x_1}, \pm\widetilde{x_2}, \ldots, \pm\widetilde{x_q}\}$ be the minimal norming set of $\mathbb{Y}$, where $\widetilde{x_k} = (x_1^k, x_2^k, \ldots, x_n^k)$, for each $1 \leq k \leq q$. Suppose on the contrary, $\sum_{k=1}^{m} \alpha_k \widetilde{a_k}$ and $\sum_{k=1}^{m} \gamma_k \widetilde{a_k}$ are two distinct best coapproximations to $\widetilde{b} = (b_1, b_2, \ldots, b_n)$ out of $\mathbb{Y}$. Therefore, from Theorem 3.4, $\widetilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m$ and $\widetilde{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathbb{R}^m$ such that $\alpha_i \neq \gamma_i$, for some $i \in \{1, 2, \ldots, m\}$ satisfies the following relations :

$$\alpha_1 \sum_{i=1}^{n} a_i^1 x_i^p + \alpha_2 \sum_{i=1}^{n} a_i^2 x_i^p + \ldots + \alpha_m \sum_{i=1}^{n} a_i^m x_i^p = \sum_{i=1}^{n} b_i x_i^p$$

and

$$\gamma_1 \sum_{i=1}^{n} a_i^1 x_i^p + \gamma_2 \sum_{i=1}^{n} a_i^2 x_i^p + \ldots + \gamma_m \sum_{i=1}^{n} a_i^m x_i^p = \sum_{i=1}^{n} b_i x_i^p.$$

for every $p \in \{1, 2, \ldots, q\}$. It is immediate from the above two equations that

$$(\alpha_1 - \gamma_1) \sum_{i=1}^{n} a_i^1 x_i^p + (\alpha_2 - \gamma_2) \sum_{i=1}^{n} a_i^2 x_i^p + \ldots + (\alpha_m - \gamma_m) \sum_{i=1}^{n} a_i^m x_i^p = 0,$$

for all $p \in \{1, 2, \ldots, q\}$. Again using Theorem 3.4, we conclude that $\sum_{k=1}^{m} (\alpha_k - \gamma_k) \widetilde{a_k}$ is a best coapproximation to $\theta \in (\ell_\infty^n)^*$ out of $\mathbb{Y}$. Therefore, $\alpha_k = \gamma_k$, for all $k \in \{1, 2, \ldots, m\}$. This contradiction completes the proof. □

Following Theorem 3.10, it is immediate that any coproximinal subspace $\mathbb{Y}$ of ℓ_1^n is also a co-Chebyshev subspace. We are now going to characterize the coproximinal(co-Chebyshev) subspaces with the help of Theorem 3.4. We first prove the following proposition.

Proposition 3.3. *Let $\mathbb{Y}$ be a subspace of ℓ_1^n and let $\mathcal{A} = \{\widetilde{a_1}, \widetilde{a_2}, \ldots, \widetilde{a_m}\}$ be a basis of $\mathbb{Y}$ with $\mathcal{Z}_{\mathcal{A}} = \emptyset$. Suppose that there are d number of nonequivalent components. Then there exist at most d number of linearly independent elements in the minimal norming set of $\mathbb{Y}$.*

Proof. Let $\{\pm\widetilde{x_1}, \pm\widetilde{x_2}, \ldots, \pm\widetilde{x_q}\}$ be the minimal norming set of $\mathbb{Y}$, where $\widetilde{x_k} = (x_1^k, x_2^k, \ldots, x_n^k)$, for each $1 \leq k \leq q$. Let $U = (u_{ij})_{1 \leq i \leq q, 1 \leq j \leq n}$ such that $u_{ij} = x_j^i$. If the r-th position and the s-th position are equivalent then from the description of $\widetilde{x_k}$, defined in the Theorem 3.2 it is easy to observe that $(x_r^1, x_r^2, \ldots, x_r^m) = \pm(x_s^1, x_s^2, \ldots, x_s^m)$. Therefore, it is easy to observe that $rank(U) \leq d$. In other words, there exist at most d number of linearly independent elements in the norming set of $\mathbb{Y}$. □

Theorem 3.11. *Let $\mathbb{Y}$ be a subspace of ℓ_1^n and let $\mathcal{A} = \{\widetilde{a_1}, \widetilde{a_2}, \ldots, \widetilde{a_m}\}$ be a basis of $\mathbb{Y}$ with $\mathcal{Z}_{\mathcal{A}} = \emptyset$. Suppose that $\mathcal{N}$ is a minimal norming set of $\mathbb{Y}$ and $\dim(span\ \mathcal{N}) = q$. Then $\mathbb{Y}$ is a coproximinal subspace if and only if $q = m$.*

Proof. Suppose that $\{\widetilde{x_1}, \widetilde{x_2}, \ldots, \widetilde{x_q}\}$ is a linearly independent set in $\mathcal{N}$, where $\widetilde{x_k} = (x_1^k, x_2^k, \ldots, x_n^k)$, where $1 \leq k \leq q$. Let $T \in \mathbb{L}(\mathbb{R}^m, \mathbb{R}^q)$ be a linear operator defined by

$$T(\widetilde{\alpha}) = \left(\sum_{j=1}^{m} \alpha_j (\sum_{i=1}^{n} a_i^j x_i^1), \sum_{j=1}^{m} \alpha_j (\sum_{i=1}^{n} a_i^j x_i^2), \ldots, \sum_{j=1}^{m} \alpha_j (\sum_{i=1}^{n} a_i^j x_i^q) \right),$$

where $\widetilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m$. Whenever $T(\alpha_1, \alpha_2, \ldots, \alpha_m) = 0$ then it is easy to observe that $\sum_{k=1}^{m} \alpha_k \widetilde{a_k}$ is the best coapproximation to $\theta \in (\ell_\infty^n)^*$ out of $\mathbb{Y}$. Clearly, $(\alpha_1, \alpha_2, \ldots, \alpha_m) =$

$\theta \in \mathbb{R}^m$ and therefore, $\ker T = \theta \in \mathbb{R}^m$. In other words, $q \geq m$. To prove the necessary part of the theorem we only need to show $q \leq m$. Let us now take $(u_1, u_2, \ldots, u_q) \in \mathbb{R}^q$. Then we choose $\widetilde{b} = (b_1, b_2, \ldots, b_n) \in \ell_1^n$ such that $\sum_{i=1}^n b_i x_i^p = u_p$, for any $1 \leq p \leq q$. Since $\{\widetilde{x_1}, \widetilde{x_2}, \ldots, \widetilde{x_q}\}$ is linearly independent, the existence of such $\widetilde{b}$ is always guaranteed. As $\mathbb{Y}$ is coproximinal, following Theorem 3.4 we obtain that for any $\widetilde{b} \in \ell_1^n$, there exists $\widetilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m$ satisfying

$$\alpha_1 \sum_{i=1}^n a_i^1 x_i^p + \alpha_2 \sum_{i=1}^n a_i^2 x_i^p + \ldots + \alpha_m \sum_{i=1}^n a_i^m x_i^p = \sum_{i=1}^n b_i x_i^p,$$

for any $p \in \{1, 2, \ldots, q\}$. Therefore, $T(\widetilde{\alpha}) = (u_1, u_2, \ldots, u_q)$, which implies that T is onto. Consequently, $q \leq m$, establishing the necessary part of the theorem.

Let us now prove the sufficient part of the theorem. Since $q = m$ and $\ker T = \theta \in \mathbb{R}^m$, it is immediate that T is invertible. This implies that for any $\widetilde{b} \in \ell_1^n$, there exists $\widetilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m$ satisfying

$$T(\widetilde{\alpha}) = \Big(\sum_{i=1}^n b_i x_i^1, \sum_{i=1}^n b_i x_i^2, \ldots, \sum_{i=1}^n b_i x_i^q \Big),$$

which implies

$$\alpha_1 \sum_{i=1}^n a_i^1 x_i^p + \alpha_2 \sum_{i=1}^n a_i^2 x_i^p + \ldots + \alpha_m \sum_{i=1}^n a_i^m x_i^p = \sum_{i=1}^n b_i x_i^p,$$

for every $p \in \{1, 2, \ldots, q\}$. Since $\{\widetilde{x_1}, \widetilde{x_2}, \ldots, \widetilde{x_q}\}$ is a basis of *span* $\mathcal{N}$, using Theorem 3.4 we conclude that $\mathbb{Y}$ is a coproximinal subspace of ℓ_1^n. $\square$

To obtain the complete characterization for coproximinal subspaces in ℓ_1^n, we now only need to consider the cases where 'the zero set' is non-empty.

Theorem 3.12. *Let $\mathbb{Y}$ be a subspace of ℓ_1^n and let $\mathcal{A} = \{\widetilde{a_1}, \widetilde{a_2}, \ldots, \widetilde{a_m}\}$ be a basis of $\mathbb{Y}$ with $|\mathcal{Z}_{\mathcal{A}}| = r > 0$. Let $n - r = k$. Then $\mathbb{Y}$ is a coproximinal subspace of ℓ_1^n if and only if $\sigma(\mathbb{Y}) = span\{\sigma(\widetilde{a_1}), \sigma(\widetilde{a_2}), \ldots, \sigma(\widetilde{a_m})\}$ is a coproximinal subspace of ℓ_1^k.*

Proof. Let us first prove the necessary part of the theorem. For any $\widetilde{w} \in \ell_1^k$, we choose $\widetilde{b} \in \ell_1^n$ such that $\sigma(\widetilde{b}) = \widetilde{w}$. Since $\mathbb{Y}$ is a coproximinal subspace of ℓ_1^n, there exist $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ such that $\sum_{i=1}^m \alpha_i \widetilde{a_i}$ is a best coapproximation to $\rho(\widetilde{b})$ out of $\mathbb{Y}$. Therefore, for any $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$,

$$\| \sum_{i=1}^m \beta_i \sigma(\widetilde{a_i}) - \sum_{i=1}^m \alpha_i \sigma(\widetilde{a_i}) \| = \| \sum_{i=1}^m \beta_i \widetilde{a_i} - \sum_{i=1}^m \alpha_i \widetilde{a_i} \| \leq \| \sum_{i=1}^m \beta_i \widetilde{a_i} - \rho(\widetilde{b}) \|$$

Now $\rho\big(\sum_{i=1}^m \beta_i\widetilde{a_i} - \rho(\widetilde{b})\big) = \sum_{i=1}^m \beta_i\widetilde{a_i} - \rho(\widetilde{b})$, so we have

$$\begin{aligned} \|\left(\sum_{i=1}^m \beta_i\widetilde{a_i} - \rho(\widetilde{b})\right)\| = \|\sigma\left(\sum_{i=1}^m \beta_i\widetilde{a_i} - \rho(\widetilde{b})\right)\| &= \|\sum_{i=1}^m \beta_i\sigma(\widetilde{a_i}) - \sigma(\rho(\widetilde{b}))\| \\ &= \|\sum_{i=1}^m \beta_i\sigma(\widetilde{a_i}) - \sigma(\widetilde{b})\| \\ &= \|\sum_{i=1}^m \beta_i\sigma(\widetilde{a_i}) - \widetilde{w}\|. \end{aligned}$$

Therefore,

$$\|\sum_{i=1}^m \beta_i\sigma(\widetilde{a_i}) - \sum_{i=1}^m \alpha_i\sigma(\widetilde{a_i})\| \leq \|\sum_{i=1}^m \beta_i\sigma(\widetilde{a_i}) - \widetilde{w}\|.$$

In other words, $\sum_{i=1}^m \alpha_i\sigma(\widetilde{a_i})$ is a best coapproximation to $\widetilde{w}$ out of $\sigma(\mathbb{Y})$. This establishes the necessary part of the theorem.

To prove the sufficient part, let $\widetilde{\eta} \in \ell_1^n$. Since $\sigma(\mathbb{Y})$ is a coproximinal subspace, there exist $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ such that $\sum_{i=1}^m \alpha_i\sigma(\widetilde{a_i})$ is a best coapproximation to $\sigma(\widetilde{\eta})$ out of $\sigma(\mathbb{Y})$. Therefore, for any $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$,

$$\begin{aligned} \|\sum_{i=1}^m \beta_i\widetilde{a_i} - \sum_{k=1}^m \alpha_i\widetilde{a_i}\| &= \|\sum_{i=1}^m \beta_i\sigma(\widetilde{a_i}) - \sum_{i=1}^m \alpha_i\sigma(\widetilde{a_i})\| \\ &\leq \|\sum_{i=1}^m \beta_i\sigma(\widetilde{a_i}) - \sigma(\widetilde{\eta})\| \\ &\leq \|\sum_{i=1}^m \beta_i\widetilde{a_i} - \widetilde{\eta}\|. \end{aligned}$$

Therefore, $\sum_{i=1}^m \alpha_i\widetilde{a_i}$ is a best coapproximation to $\widetilde{\eta}$ out of $\mathbb{Y}$. This completes the theorem. □

Our final result in this section reads as follows.

Theorem 3.13. *Let $\mathbb{Y}$ be a subspace of ℓ_1^n and let $\mathcal{A} = \{\widetilde{a_1}, \widetilde{a_2}, \ldots, \widetilde{a_m}\}$ be a basis of $\mathbb{Y}$ with $|\mathcal{Z}_{\mathcal{A}}| = r > 0$. Then $\mathbb{Y}$ is not a co-Chebyshev subspace of ℓ_1^n.*

Proof. Let $\widetilde{b} = (b_1, b_2, \ldots, b_n) \notin \mathbb{Y}$ be such that $\rho(\widetilde{b}) = \theta \in \ell_1^n$, which implies that $b_i = 0$, for any $i \notin \mathcal{Z}_{\mathcal{A}}$. Now for any $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ satisfying $\|\sum_{i=1}^m \alpha_i\widetilde{a_i}\| \leq \|\widetilde{b}\|$ and for any $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$, it is easy to observe that

$$\begin{aligned} \|\sum_{i=1}^m \beta_i\widetilde{a_i} - \sum_{i=1}^m \alpha_i\widetilde{a_i}\| &\leq \|\sum_{i=1}^m \beta_i\widetilde{a_i}\| + \|\sum_{i=1}^m \alpha_i\widetilde{a_i}\| \\ &\leq \|\sum_{i=1}^m \beta_i\widetilde{a_i}\| + \|\widetilde{b}\|. \end{aligned}$$

We also note that

$$\begin{aligned}
\|\sum_{i=1}^{m}\beta_i\widetilde{a_i}-\widetilde{b}\| &= \sum_{j=1}^{n}\left|\left(\sum_{i=1}^{m}\beta_i a_j^i - b_j\right)\right| \\
&= \sum_{j\notin\mathcal{Z}_{\mathcal{A}}}\left|\left(\sum_{i=1}^{m}\beta_i a_j^i - b_j\right)\right| + \sum_{j\in\mathcal{Z}_{\mathcal{A}}}\left|\left(\sum_{i=1}^{m}\beta_i a_j^i - b_j\right)\right| \\
&= \sum_{j\notin\mathcal{Z}_{\mathcal{A}}}\left|\sum_{i=1}^{m}\beta_i a_j^i\right| + \sum_{j\in\mathcal{Z}_{\mathcal{A}}}|b_j| \\
&= \|\sum_{i=1}^{m}\beta_i\widetilde{a_i}\| + \|\widetilde{b}\|.
\end{aligned}$$

Therefore,

$$\|\sum_{i=1}^{m}\beta_i\widetilde{a_i}-\sum_{i=1}^{m}\alpha_i\widetilde{a_i}\| \leq \|\sum_{i=1}^{m}\beta_i\widetilde{a_i}-\widetilde{b}\|.$$

In other words, for any $\alpha_1,\alpha_2,\ldots,\alpha_m \in \mathbb{R}$ such that $\|\sum_{i=1}^{m}\alpha_i\widetilde{a_i}\| \leq \|\widetilde{b}\|$, $\sum_{i=1}^{m}\alpha_i\widetilde{a_i}$ is a best coapproximations to $\widetilde{b}$ out of $\mathbb{Y}$. Therefore, $\mathbb{Y}$ is not a co-Chebyshev subspace of ℓ_1^n. This establishes the theorem. □

We end this section with examples of both coproximinal and not coproximinal subspaces for which the zero set is non-empty.

Example 3.14. *It can be easily verified by using the methods developed in this chapter that* $\mathbb{Y}_1$ *is a coproximinal subspace of* ℓ_1^7*, whereas* $\mathbb{Y}_2$ *is not, where*

$$\mathbb{Y}_1 = span\{(1,1,2,0,4,-2,0),(1,2,2,0,4,-4,0)\},$$

$$\mathbb{Y}_2 = span\{(1,0,2,3,-1,-2,0),(-1,0,1,0,1,-1,0)\}.$$

Moreover,

$$\mathbb{Y}_3 = span\{(1,1,2,4,-2),(1,2,2,4,-4)\}$$

is a co-Chebyshev subspace of ℓ_1^5*, but* $\mathbb{Y}_1$ *is not co-Chebyshev.*

3.5 Concluding remark

The computational difficulty in solving the best coapproximation problem arises essentially from the non-linear nature of the inequalities associated with it. We have illustrated in this chapter that by applying Birkhoff-James orthogonality techniques, it is possible to reduce the much harder non-linear problem into a system of linear equations (see Theorem 3.4).

We have presented explicit examples to highlight the different possibilities for subspaces of ℓ_1^n, from the perspective of best coapproximation. Indeed, it follows from our observations that the newly introduced "zero set" of a subspace plays a fundamental role in the whole scheme of things (see Example 4.2, Corollary 3.10). We have also explored the relationship between coproximinal subspaces and co-Chebyshev subspaces of ℓ_1^n, depending on the zero sets of the concerned subspaces. In particular, it is to be noted that there exists a coproximinal subspace of ℓ_1^n, which is not co-Chebyshev (see Example 3.14).

In view of the methods developed here, applications of the concept of orthogonality in solving the best coapproximation problem in Banach spaces seem to be a promising direction of research, resulting in efficient algorithms which are computationally advantageous. We have presented several numerical examples in support of this, in the specific setting of ℓ_1^n spaces. For an analogous approach to the best coapproximation problem in ℓ_∞^n spaces, the readers are referred to the recent article [52]. As a matter of fact, it may be interesting to apply Birkhoff-James orthogonality towards obtaining computationally efficient algorithmic solutions to the said problem, in the setting of other classical Banach spaces, such as the ℓ_p^n spaces, where $1 < p < \infty$.

CHAPTER 4

SOME SPECIAL SUBSPACES FROM THE PERSPECTIVE OF BEST COAPPROXIMATION

4.1 Introduction

In the preceding chapters, we provided computational approaches to study the best coapproximations problem the space of diagonal matrices and the space ℓ_1^n. In particular, we examined coproximinal subspaces in those ambient spaces, which represent the ideal case or best case from the perspective of existence of best coapproximation. In contrast, this chapter introduces and explores two new notions- *anti-coproximinal subspace* and *strongly anticoproximinal subspaces* which may regard as the least favorable candidates for the purpose of the best coapproximation problem. Through this study, we explore the extreme nature of these two subspaces, which deepen the understanding of the theory of geometry of Banach space. Having described the motivation behind this study, let us now introduce the relevant notations and terminologies.

Content of this chapter is based on the following paper:

- Sohel, S., Ghosh, S., Sain, D., Paul, K., *Some special subspaces from the perspective of best coapproximation*, Monatsh. Math., **204** (2024), 969-987.

Let $\mathbb{X}, \mathbb{Y}$ denote real Banach spaces and let $\mathbb{H}$ denote a real Hilbert space. We use the notations $B_{\mathbb{X}}$ and $S_{\mathbb{X}}$, for the unit ball $B_{\mathbb{X}} := \{x \in \mathbb{X} : \|x\| \leq 1\}$ and the unit sphere $S_{\mathbb{X}} := \{x \in \mathbb{X} : \|x\| = 1\}$ of $\mathbb{X}$, respectively. The dual space of $\mathbb{X}$ is denoted by $\mathbb{X}^*$. The annihilator of a subspace $\mathbb{Y}$ of $\mathbb{X}$ is defined as $\mathbb{Y}^{\perp} := \{f \in \mathbb{X}^* : f(y) = 0, \text{ for each } y \in \mathbb{Y}\}$. For a subspace $\mathbb{Z}$ of $\mathbb{X}^*$, the pre-annihilator of $\mathbb{Z}$ is defined as ${}^{\perp}\mathbb{Z} := \{x \in \mathbb{X} : f(x) = 0, \text{ for each } f \in \mathbb{Z}\}$. For a non-empty convex subset A of $\mathbb{X}$, an element $z \in A$ is said to be an extreme point of A if whenever $z = (1-t)x + ty$, for some $t \in (0,1)$ and some $x, y \in A$, then $x = y = z$. The collection of all the extreme points of A is denoted as $Ext(A)$. A Banach space $\mathbb{X}$ is said to be strictly convex if $Ext(B_{\mathbb{X}}) = S_{\mathbb{X}}$. It is easy to observe that strict convexity of $\mathbb{X}$ is equivalent to the geometric condition that $S_{\mathbb{X}}$ does not contain any non-trivial straight line segment. We recall that a point $x \in S_{\mathbb{X}}$ is said to be a rotund point [16] of $B_{\mathbb{X}}$ if $\|y\| = \|\frac{x+y}{2}\| = 1$ implies that $x = y$. Clearly, if every point of $S_{\mathbb{X}}$ is rotund then $\mathbb{X}$ is strictly convex. Given any non-zero $x \in \mathbb{X}$, $f \in S_{\mathbb{X}^*}$ is said to be a support functional of x if $f(x) = \|x\|$. The set of all support functional(s) of a nonzero $x \in \mathbb{X}$ is written as $J(x) := \{f \in S_{\mathbb{X}^*} : f(x) = \|x\|\}$. For any nonzero $x \in \mathbb{X}$, x is said to be smooth if $J(x)$ is singleton. A Banach space $\mathbb{X}$ is said to be smooth if each $x \in S_{\mathbb{X}}$ is smooth. The collection of all smooth points in $\mathbb{X}$ is denoted by $Sm(\mathbb{X})$. For a subspace $\mathbb{Y}$ of $\mathbb{X}$, let

$$\mathcal{J}_{\mathbb{Y}} = \{f \in S_{\mathbb{X}^*} : f(y) = 1, \text{for some } y \in Sm(\mathbb{X}) \cap S_{\mathbb{Y}}\}.$$

It is easy to check that $\mathcal{J}_{\mathbb{Y}} \subseteq Ext(B_{\mathbb{X}^*})$. Whenever $Sm(\mathbb{X}) \cap S_{\mathbb{Y}} = \emptyset$, we define $\mathcal{J}_{\mathbb{Y}} = \emptyset$. Let us recall the well known definition of the modulus of smoothness of a Banach space $\mathbb{X}(\neq \{0\})$, which is denoted $\rho_{\mathbb{X}}(t)$, and is defined as follows:

$$\rho_{\mathbb{X}}(t) \quad = \quad \sup\left\{\frac{\|x+ty\| + \|x-ty\|}{2} - 1 : x, y \in S_{\mathbb{X}}\right\},$$

where $t \in (0, +\infty)$. The space $\mathbb{X}(\neq \{0\})$ is said to be uniformly smooth ([27, Def. 5.5.2]) if $\frac{\rho_{\mathbb{X}}(t)}{t} \to 0$, as $t \to 0^+$.

Given any $f \in \mathbb{X}^*$, the kernel of f is denoted by $\ker f := \{x \in \mathbb{X} : f(x) = 0\}$. A finite-dimensional Banach space $\mathbb{X}$ is said to be a polyhedral Banach space if $Ext(B_{\mathbb{X}})$ is finite. A convex set $F \subset S_{\mathbb{X}}$ is said to be a face of $B_{\mathbb{X}}$ if for some $x_1, x_2 \in S_{\mathbb{X}}$, $(1-t)x_1 + tx_2 \in F$ implies that $x_1, x_2 \in F$, where $0 < t < 1$. A maximal face of $B_{\mathbb{X}}$ is said to be a facet. We use the notation $int(F)$ to denote the interior of a face F endowed with the usual subspace topology of F. Given a subset M of $\mathbb{X}^*$, $\overline{M}^{w^*}$ denotes the closure of M with respect to the weak*- topology defined on $\mathbb{X}^*$. We also recall that $\mathbb{X}$ satisfies the Kadets-Klee Property if whenever $\{x_n\}$ is a sequence in $\mathbb{X}$ and $x \in \mathbb{X}$ such that $x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\|$, it follows that $x_n \to x$. Given any $x \in \mathbb{X}$ and a subspace $\mathbb{Y}$ of $\mathbb{X}$, $\mathcal{R}_{\mathbb{Y}}(x)$ denotes the (possibly empty) set of all best coapproximations to x out of $\mathbb{Y}$. A subspace $\mathbb{Y}$ of the Banach space $\mathbb{X}$ is said to be coproximinal if for any $x \in \mathbb{X}$, $\mathcal{R}_{\mathbb{Y}}(x) \neq \emptyset$.

We also need the notion of approximate Birkhoff-James orthogonality in our study. In [13], Dragomir first introduced the approximate Birkhoff-James orthogonality in the following way: Let $\epsilon \in [0, 1)$. Then for $x, y \in \mathbb{X}$, x is said to be approximate ϵ-Birkhoff-James orthogonal to y if for each $\lambda \in \mathbb{R}$, the following holds:

$$\|x + \lambda y\| \geq (1 - \epsilon)\|x\|.$$

Later on Chmieliński [9] introduced another version of the approximate Birkhoff-James orthogonality. Let $\epsilon \in [0, 1)$. Given any $x, y \in \mathbb{X}$, we say that x is said to be approximate orthogonal to y, written as $x \perp_B^{\epsilon} y$, if

$$\|x + \lambda y\|^2 \geq \|x\|^2 - 2\epsilon\|x\|\|\lambda y\|.$$

Recently [10], an equivalent definition of the approximate orthogonality has been obtained:

$$x \perp_B^{\epsilon} y \iff \|x + \lambda y\| \geq \|x\| - \epsilon\|\lambda y\|, \text{ for every } \lambda \in \mathbb{R}.$$

4.2 Anti-coproximinal subspace

4.2.1 Definition and basic properties

The primary purpose of this chapter is to investigate the least favorable scenario that can arise in studying the best coapproximation problem. Accordingly, we introduce the following type of subspaces of a Banach space from the perspective of best coapproximation.

Definition 4.1. *Let $\mathbb{Y}$ be a subspace of Banach space $\mathbb{X}$. Then $\mathbb{Y}$ is said to be an anti-coproximinal subspace of $\mathbb{X}$ if for any given $x \in \mathbb{X} \setminus \mathbb{Y}$, there does not exist any best coapproximation to x out of $\mathbb{Y}$.*

Applying the connection between best coapproximation and Birkhoff-James orthogonality, it is quite easy to verify that $\mathbb{Y}$ is an anti-coproximinal subspace of $\mathbb{X}$ if and only if there exists no $x \in \mathbb{X} \setminus \mathbb{Y}$ such that $\mathbb{Y} \perp_B x$.

We note that if a subspace $\mathbb{Y}$ is not coproximinal in $\mathbb{X}$ then it is not necessarily true that $\mathbb{Y}$ is an anti-coproximinal subspace of $\mathbb{X}$. To justify this observe the example given in [52, Example 2.13]. It should be noted that every one-dimensional subspace is coproximinal in any Banach space [15, Lemma 1]. Whenever the anti-coproximinal subspaces are concerned, we only consider the proper closed subspaces of dimension strictly greater than one.

In the following proposition, we observe that every dense subspace of a Banach space is strongly anti-coproximinal in that space and consequently, anti-coproximinal too.

Proposition 4.1. *Let $\mathbb{Y}$ be a dense subspace of a Banach space $\mathbb{X}$. Then $\mathbb{Y}$ is an anti-coproximinal subspace of $\mathbb{X}$.*

Proof. Suppose on the contrary that $\mathbb{Y}$ is not an anti-coproximinal subspace of $\mathbb{X}$. Then there exists an $x \in \mathbb{X} \setminus \mathbb{Y}$ such that $\mathbb{Y} \perp_B x$. Since $\mathbb{Y}$ is dense in $\mathbb{X}$, it follows that there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{Y}$ such that $y_n \to x$. Also, we note that $y_n \perp_B x$. Then for each $\lambda \in \mathbb{R}$, we have $\|y_n + \lambda x\| \geq \|y_n\|$. Since the norm function is continuous, letting $n \to \infty$, we get that $\|x + \lambda x\| \geq \|x\|$, for each $\lambda \in \mathbb{R}$. This implies that $x \perp_B x$, and therefore $x = 0$, which contradicts the fact that $x \in \mathbb{X} \setminus \mathbb{Y}$. □

It is easy to observe that if $\mathbb{Y}$ is anti-coproximinal in $\mathbb{X}$ and $\mathbb{Z}$ is a subspace of $\mathbb{X}$ containing $\mathbb{Y}$ then $\mathbb{Y}$ is anti-coproximinal in $\mathbb{Z}$. However, it is quite obvious that if $\mathbb{Y}$ is anti-coproximinal in $\mathbb{Z}$, it does not imply that $\mathbb{Y}$ is anti-coproximinal in $\mathbb{X}$ (see Remark 4.9). Therefore, it is important to specify the mother space whenever we consider the anti-coproximinal.

4.2.2 Anti-coproximinal subspaces in smooth Banach space

In the next theorem we characterize the anti-comproximinal subspaces in a smooth Banach space.

Theorem 4.1. *Let $\mathbb{Y}$ be a subspace of a smooth Banach space $\mathbb{X}$. Then $\mathbb{Y}$ is an anti-coproximinal subspace of $\mathbb{X}$ if and only if $\overline{span\ \mathcal{J}_{\mathbb{Y}}}^{w^*} = \mathbb{X}^*$.*

Proof. Let us first prove the sufficient part of the theorem. Suppose on the contrary that $\mathbb{Y}$ is not an anti-coproximinal subspace of $\mathbb{X}$. This implies that for some $x \in \mathbb{X} \setminus \mathbb{Y}$, there exists a $y_0 \in \mathbb{Y}$ such that y_0 is a best coapproximation to x out of $\mathbb{Y}$. Since $\mathbb{X}$ is smooth, it follows from Theorem 1.12 that for any $y \in \mathbb{Y}$, $f_y(x - y_0) = 0$, where $J(y) = \{f_y\}$. It is easy to see that $\mathcal{J}_{\mathbb{Y}} = \{f_y : y \in \mathbb{Y}\}$. This implies, $g(x - y_0) = 0$, for any $g \in span\ \mathcal{J}_{\mathbb{Y}}$. Now let us consider $f \in \mathbb{X}^*$. Since $\overline{span\ \mathcal{J}_{\mathbb{Y}}}^{w^*} = \mathbb{X}^*$, it follows that there exists a net $\{f_\alpha\}_{\alpha\in\Lambda} \subset span\ \mathcal{J}_{\mathbb{Y}}$ such that f_α is weak*-convergent to f. Since $f_\alpha(x - y_0) = 0$, for each $\alpha \in \Lambda$, it follows that $f(x - y_0) = 0$. Note that $f \in \mathbb{X}^*$ is taken arbitrarily. Thus we obtain that $f(x - y_0) = 0$, for all $f \in \mathbb{X}^*$, i.e., $x - y_0 = 0$, which contradicts the fact that $x \in \mathbb{X} \setminus \mathbb{Y}$.

Now we prove the necessary part of the theorem. Suppose on the contrary that $\overline{span\ \mathcal{J}_{\mathbb{Y}}}^{w^*} \subsetneq \mathbb{X}^*$. Clearly, ${}^{\perp}(span\ \mathcal{J}_{\mathbb{Y}}) = \cap_{f\in\mathcal{J}_{\mathbb{Y}}} \ker f$. We note from [41, Th. 4.7] that $({}^{\perp}(span\ \mathcal{J}_{\mathbb{Y}}))^{\perp} = \overline{span\ \mathcal{J}_{\mathbb{Y}}}^{w^*}$. This implies that $(\cap_{f\in\mathcal{J}_{\mathbb{Y}}} \ker f)^{\perp} = \overline{span\ \mathcal{J}_{\mathbb{Y}}}^{w^*}$. Following [41, Th. 4.7], we obtain that $(\cap_{f\in\mathcal{J}_{\mathbb{Y}}} \ker f)^*$ is isometrically isomorphic to $\mathbb{X}^*/\overline{span\ \mathcal{J}_{\mathbb{Y}}}^{w^*}$. Therefore, $(\cap_{f\in\mathcal{J}_{\mathbb{Y}}} \ker f)^* \neq 0$, which implies that $\cap_{f\in\mathcal{J}_{\mathbb{Y}}} \ker f \neq 0$. Moreover, it is straightforward to see that $(\cap_{f\in\mathcal{J}_{\mathbb{Y}}} \ker f) \cap \mathbb{Y} = \{0\}$. So, there exists a $z \in \mathbb{X}\setminus\mathbb{Y}$ such that $z \in \cap_{f\in\mathcal{J}_{\mathbb{Y}}} \ker f$. Since $\mathbb{X}$ is smooth, applying [20, Th. 2.1], we obtain that $\mathbb{Y} \perp_B z$. Thus 0 is a best coapproximation to z out of $\mathbb{Y}$. This contradicts the fact that $\mathbb{Y}$ is an anti-coproximinal subspace of $\mathbb{X}$. This proves the necessary part and completes the proof of the theorem. □

If $\mathbb{X}$ is finite-dimensional then we have the following corollary.

Corollary 4.1. *Let $\mathbb{Y}$ be a subspace of an n-dimensional smooth Banach space $\mathbb{X}$. Then $\mathbb{Y}$ is an anti-coproximinal subspace of $\mathbb{X}$ if and only if $dim(span\ \mathcal{J}_{\mathbb{Y}}) = n$.*

Next we give an example of an anti-coproximinal subspace in ℓ_p^n, where $2 < p < \infty$. We require the following well-known result which explicitly describes the support functional of an element of ℓ_p^n. Suppose that ϕ is the isometric isomorphism between $(\ell_p^n)^*$ and ℓ_q^n, where $\frac{1}{p}+\frac{1}{q}=1$.

Lemma 4.1. *Let $\widetilde{x} = (x_1, x_2, \ldots, x_n) \in \ell_p^n$, where $1 < p < \infty$. Then $J(\widetilde{x}) = \{\widetilde{f}\}$, where $\phi(\widetilde{f}) = \Big(\frac{x_1|x_1|^{p-2}}{\|\widetilde{x}\|_p^{p/q}}, \frac{x_2|x_2|^{p-2}}{\|\widetilde{x}\|_p^{p/q}}, \ldots, \frac{x_n|x_n|^{p-2}}{\|\widetilde{x}\|_p^{p/q}}\Big) \in \ell_q^n$, $\frac{1}{p}+\frac{1}{q}=1$ and $\|\widetilde{x}\|_p = \big(\sum_{i=1}^n |x_i|^p\big)^{\frac{1}{p}}$.*

Example 4.2. *Let us consider the space ℓ_p^n, where $p \in (1,\infty)\setminus\{2\}$ and $n \geq 3$ with $\{e_1, e_2, \ldots, e_n\}$ as the standard ordered basis. Suppose that $\mathbb{Y}$ is a hyperspace of ℓ_p^n, where $\mathbb{Y} = span\{\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_{n-1}\}$, $\widetilde{x}_1 = (1,1,1,0,\ldots,0), \widetilde{x}_2 = (1,2,3,0,\ldots,0), \widetilde{x}_3 = e_4, \ldots, \widetilde{x}_{n-1} = e_n$. Let $J(\widetilde{x}_i) = \{\widetilde{f}_i\}$, for any $1 \leq i \leq n$. Clearly, $\widetilde{f}_i \in \mathcal{J}_{\mathbb{Y}}$. Applying Lemma 4.1, we obtain the following:*
$\phi(\widetilde{f}_1) = \left(\frac{1}{3^{1-\frac{1}{p}}}, \frac{1}{3^{1-\frac{1}{p}}}, \frac{1}{3^{1-\frac{1}{p}}}, 0, \ldots, 0\right) \in \ell_q^n$,
$\phi(\widetilde{f}_2) = \left(\frac{1}{(1+2^p+3^p)^{1-\frac{1}{p}}}, \frac{2^{p-1}}{(1+2^p+3^p)^{1-\frac{1}{p}}}, \frac{3^{p-1}}{(1+2^p+3^p)^{1-\frac{1}{p}}}, 0, \ldots, 0\right) \in \ell_q^n$,
and $\phi(\widetilde{f}_k) = e_{k+1} \in \ell_q^n$, for all $3 \leq k \leq n-1$. Consider the element $\widetilde{x}_n = 3\widetilde{x}_1 - \widetilde{x}_2 \in \mathbb{Y}$ and let $J(\widetilde{x}_n) = \{\widetilde{f}_n\}$. Again using Lemma 4.1, we get $\phi(\widetilde{f}_n) = \left(\frac{2^{p-1}}{(2^p+1)^{1-\frac{1}{p}}}, \frac{1}{(2^p+1)^{1-\frac{1}{p}}}, 0, \ldots, 0\right) \in \ell_q^n$. A straightforward computation shows that $\{\widetilde{f}_1, \widetilde{f}_2, \ldots, \widetilde{f}_n\}$ is a linearly independent set. Therefore, $dim(span\ \mathcal{J}_{\mathbb{Y}}) = n$. Thus from Corollary 4.1, we conclude that $\mathbb{Y}$ is an anti-coproximinal subspace of ℓ_p^n, where $p \in (1,\infty)\setminus\{2\}$. By a similar computation, it can be shown that $\mathbb{W} = \underbrace{\mathbb{Y}\oplus\mathbb{Y}\oplus\ldots\oplus\mathbb{Y}}_{r-times}$ is an anti-coproximinal subspace of ℓ_p^{rn}.

It is easy to note from Example 4.2 that for every $p \in (1,\infty) \setminus \{2\}$ and for every $n > 2$, there exists an anti-coproximinal subspace of ℓ_p^n. Combining with the well known fact that *every closed subspace of a Hilbert space is coproximinal* ([15, Lemma. 1]), we can characterize the Hilbert space ℓ_2^n among the ℓ_p^n spaces.

Theorem 4.3. *Let $\mathbb{X} = \ell_p^n$, where $1 < p < \infty$ and $n \geq 3$. Then $p = 2$ if and only if there does not exist any anti-coproximinal subspace in $\mathbb{X}$.*

The previous result can be further extended to characterize the Hilbert space among smooth Banach spaces having dimension at least 3.

Theorem 4.4. *Let $\mathbb{X}$ be a smooth Banach space and let $dim(\mathbb{X}) \geq 3$. Then $\mathbb{X}$ is Hilbert space if and only if there does not exist any anti-coproximinal closed hyperspace in $\mathbb{X}$.*

Proof. Since the necessary part follows from [15, Th. 1], we only prove the sufficient part. Suppose that $\mathbb{Y}$ is a closed hyperspace of $\mathbb{X}$. Since $\mathbb{Y}$ is not an anti-coproximinal subspace of $\mathbb{X}$, it follows from Theorem 4.1 that $\overline{span\ \mathcal{J}_{\mathbb{Y}}}^{w^*} \subsetneq \mathbb{X}^*$. Now following similar arguments as given in the proof of the necessary part of Theorem 4.1, we observe that there exists an element $z \in \mathbb{X} \setminus \mathbb{Y}$ such that $\mathbb{Y} \perp_B z$. Following [19, Th. 4], we conclude that $\mathbb{X}$ is a Hilbert space. □

Remark 4.5. *We recall that every one-dimensional subspace of a Banach space is coproximinal. Therefore, it is clear that the previous result is not valid for $n = 2$.*

4.2.3 Anti-coproximinal subspaces in polyhedral Banach space

We begin with the following characterization of the anti-coproximinal subspaces of a finite-dimensional polyhedral Banach space, provided that $Sm(\mathbb{X}) \cap \mathbb{Y}$ is dense in $\mathbb{Y}$.

Theorem 4.6. *Let $\mathbb{Y}$ be a subspace of an n-dimensional polyhedral Banach space $\mathbb{X}$ such that $Sm(\mathbb{X}) \cap \mathbb{Y}$ is dense in $\mathbb{Y}$. Then $\mathbb{Y}$ is an anti-coproximinal subspace of $\mathbb{X}$ if and only if there are n-linearly independent elements in $\mathcal{J}_{\mathbb{Y}}$.*

Proof. We first prove the sufficient part. Suppose on the contrary that $\mathbb{Y}$ is not an anti-coproximinal subspace of $\mathbb{X}$. Then there exists an element $x \in \mathbb{X} \setminus \mathbb{Y}$ and a $y_0 \in \mathbb{Y}$ such that $y_0 \in \mathcal{R}_{\mathbb{Y}}(x)$. Therefore, from Theorem 1.12, it follows that $x - y_0 \in \ker f$, for all $f \in \mathcal{J}_{\mathbb{Y}}$. Since $\mathcal{J}_{\mathbb{Y}}$ contains n linearly independent elements and $\dim(\mathbb{X}^*) = n$, it follows that $\cap_{f \in \mathcal{J}_{\mathbb{Y}}} \ker f = \{0\}$. Therefore, $x - y_0 = 0$, i.e., $x = y_0$, which is a contradiction. This completes the proof of the sufficient part.

To prove the necessary part, we first show that for any $y \in S_{\mathbb{Y}}$, $\mathcal{J}_{\mathbb{Y}} \cap J(y) \neq \emptyset$. Since $Sm(\mathbb{X}) \cap \mathbb{Y}$ is dense in $\mathbb{Y}$, it is immediate that $Sm(\mathbb{X}) \cap S_{\mathbb{Y}}$ is dense in $S_{\mathbb{Y}}$, and therefore, for any $y \in S_{\mathbb{Y}}$, there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset Sm(\mathbb{X}) \cap S_{\mathbb{Y}}$ such that $y_n \to y$. Suppose that $J(y_n) = \{f_n\} \subset \mathcal{J}_{\mathbb{Y}}$, for each $n \in \mathbb{N}$. Therefore, it is easy to see that $f_n(y) \to 1$. Since $\mathbb{X}$ is polyhedral and $\mathcal{J}_{\mathbb{Y}} \subset Ext(B_{\mathbb{X}^*})$, it follows that $\mathcal{J}_{\mathbb{Y}}$ is finite. Moreover, since for each $n \in \mathbb{N}$, $f_n \in \mathcal{J}_{\mathbb{Y}}$ and $\mathcal{J}_{\mathbb{Y}}$ is finite, we get that for some $f \in \mathcal{J}_{\mathbb{Y}}$, $f(y) = 1$. Thus $f \in \mathcal{J}_{\mathbb{Y}} \cap J(y)$. If possible, suppose that $\mathcal{J}_{\mathbb{Y}}$ contains exactly k linearly independent elements such that $k < n$. Let us assume that $(\neq 0)x \in \cap_{f\in\mathcal{J}_{\mathbb{Y}}} \ker f$. Clearly, $x \notin \mathbb{Y}$. Otherwise, from the above arguments $f(x) = 1$, for some $f \in \mathcal{J}_{\mathbb{Y}}$. Since for each $y \in S_{\mathbb{Y}}$, $\mathcal{J}_{\mathbb{Y}} \cap J(y) \neq \emptyset$, it follows that there exists an $f \in J(y)$ such that $f(x) = 0$. It is immediate from Theorem 1.12 that 0 is a best coapproximation to x out of $\mathbb{Y}$, which is not possible since $\mathbb{Y}$ is an anti-coproximinal subspace of $\mathbb{X}$. This completes the proof of the necessary part. Hence the theorem. □

We note that the sufficient part of the above theorem holds true for any finite-dimensional Banach space $\mathbb{X}$ without any additional assumptions. In particular, this implies that *if $\mathbb{Y}$ is a subspace of an n-dimensional Banach space $\mathbb{X}$ such that there are n-linearly independent elements in $\mathcal{J}_{\mathbb{Y}}$, then $\mathbb{Y}$ is an anti-coproximinal subspace of $\mathbb{X}$.*

Although we may not characterize the anti-coproximinal subspaces in finite-dimensional polyhedral Banach spaces, we can characterize this notions in the space ℓ_1^n and ℓ_∞^n. We fist study the anti-coproximinal subspaces of ℓ_∞^n. The best coapproximation problem in ℓ_∞^n was studied in Chapter 2 using the concept of the $*$-Property which plays a crucial role in the whole scheme of things. For the convenience of the readers, let us recall the definition of the $*$-Property.

Definition 4.2. *[52] Let $\mathcal{A} = \{\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_m\}$ be a set of linearly independent elements in ℓ_∞^n, where $1 \leq m \leq n$ and $\widetilde{a}_k = (a_1^k, a_2^k, \ldots, a_n^k)$, for each $1 \leq k \leq m$.*

(i) For each $i \in \{1, 2, \ldots, n\}$, the i-th component of $\mathcal{A}$ is defined as $(a_i^1, a_i^2, \ldots, a_i^m)$.

(ii) The positively associative set $P_i^+(\mathcal{A})$ of the i-th component is defined as:

$$P_i^+(\mathcal{A}) := \{j \in \{1, 2, \ldots, n\} : (a_i^1, a_i^2, \ldots, a_i^m) = (a_j^1, a_j^2, \ldots, a_j^m)\}.$$

Similarly, the negatively associated set $P_i^-(\mathcal{A})$ is defined as:

$$P_i^-(\mathcal{A}) := \{j \in \{1, 2, \ldots, n\} : (a_i^1, a_i^2, \ldots, a_i^m) = -(a_j^1, a_j^2, \ldots, a_j^m)\}.$$

We write $P_i^+(\mathcal{A}) = P_i^+$ and $P_i^-(\mathcal{A}) = P_i^-$.

(iii) The i-th component of $\mathcal{A}$ is said to satisfy the $*$-Property if there exist $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$

such that the following holds true:

$$\Big|\sum_{k=1}^{m}\beta_k a_i^k\Big| > \max\Big\{\Big|\sum_{k=1}^{m}\beta_k a_j^k\Big| : j\in\{1,2,\ldots,n\}\setminus P_i^+\cup P_i^-\Big\}.$$

We next recall the characterization of smooth points of the unit sphere of ℓ_∞^n and the extreme points of the unit ball of $(\ell_\infty^n)^*$.

Proposition 4.2. *$\widetilde{x}=(x_1,x_2,\ldots,x_n)\in S_{\ell_\infty^n}$ is a smooth point if and only if there exists $i_0\in\{1,2,\ldots,n\}$ such that $|x_{i_0}|=1$ and $|x_j|<1$, for each $j\in\{1,2,\ldots,n\}\setminus\{i_0\}$. Moreover, $f\in(\ell_\infty^n)^*$ is an extreme point of $B_{(\ell_\infty^n)^*}$ if and only if there exists $i_0\in\{1,2,\ldots,n\}$ such that $f(x_1,x_2,\ldots,x_n)=x_{i_0}$, for any $(x_1,x_2,\ldots,x_n)\in\ell_\infty^n$.*

The following lemma is essential to characterize the anti-coproximinal and the strongly anti-coproximinal subspaces in ℓ_∞^n.

Lemma 4.2. *Let $\mathcal{A}=\{\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_m\}$ be a set of linearly independent elements in ℓ_∞^n, where $1<m<n$ and $\widetilde{a}_k=(a_1^k,a_2^k,\ldots,a_n^k)$, for each $1\le k\le m$. Let $\mathbb{Y}=span\ \mathcal{A}$. If every component of $\mathcal{A}$ satisfies the $*$-Property and $|P_i^+\cup P_i^-|=1$, for all $1\le i\le n$ then for every $f\in Ext(B_{(\ell_\infty^n)^*})$, there exists a $\widetilde{y}\in Sm(\mathbb{X})\cap S_{\mathbb{Y}}$ such that $J(\widetilde{y})=\{f\}$.*

Proof. Let $f_i\in(\ell_\infty^n)^*$ be such that $f_i(x_1,x_2,\ldots,x_n)=x_i$. Clearly, $Ext(B_{(\ell_\infty^n)^*})=\{\pm f_1,\pm f_2,\ldots,\pm f_n\}$. Since the i-th component satisfies the $*$-Property and $|P_i^+\cup P_i^-|=1$ then there exist $\beta_1,\beta_2,\ldots,\beta_m\in\mathbb{R}$ such that

$$|\sum_{k=1}^{m}\beta_k a_i^k| > \max\Big\{|\sum_{k=1}^{m}\beta_k a_j^k| : j\in\{1,2,\ldots,n\}\setminus\{i\}\Big\}.$$

Clearly, $\|\sum_{k=1}^{m}\beta_k\widetilde{a_k}\|=|\sum_{k=1}^{m}\beta_k a_i^k|$. Take $\widetilde{y}=\frac{\sum_{k=1}^{m}\beta_k\widetilde{a_k}}{\|\sum_{k=1}^{m}\beta_k\widetilde{a_k}\|}$. Clearly, $|f_i(\widetilde{y})|=1$. It is easy to observe from Proposition 4.2 that $\widetilde{y}\in Sm(\mathbb{X})\cap S_{\mathbb{Y}}$. Thus either $J(\widetilde{y})=\{f_i\}$ or $J(-\widetilde{y})=\{f_i\}$. □

We are now ready to present the desired characterization.

Theorem 4.7. *Let $\mathcal{A}=\{\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_m\}$ be a set of linearly independent elements in ℓ_∞^n, where $1<m<n$ and $\widetilde{a}_k=(a_1^k,a_2^k,\ldots,a_n^k)$, for each $1\le k\le m$. Let $\mathbb{Y}=span\ \mathcal{A}$. Then the following statements are equivalent:*

(i) $\mathbb{Y}$ *is an anti-coproximinal subspace of* ℓ_∞^n.

(ii) Every component of $\mathcal{A}$ satisfies the $*$-Property and $|P_i^+\cup P_i^-|=1$, for all $1\le i\le n$.

Proof. First we prove (i) $\implies$ (ii). Let $f_i(x_1, x_2, \ldots, x_n) = x_i$, for any $(x_1, x_2, \ldots, x_n) \in \ell_\infty^n$. Clearly, $Ext(B_{(\ell_\infty^n)^*}) = \{\pm f_1, \pm f_2, \ldots, \pm f_n\},$. Suppose on the contrary that for some $j \in \{1, 2, \ldots, n\}$, the j-th component does not satisfy the $*$-Property. It is easy to observe that there does not exist any $y \in \mathbb{Y}$ such that $J(y) = \{f_j\}$. Otherwise, considering $y = \sum_{k=1}^m \beta_k \widetilde{a}_k$, for some $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$, we obtain that $|\sum_{k=1}^m \beta_k a_j^k| = |f_j(y)| > |f_t(y)| = |\sum_{k=1}^m \beta_k a_t^k|$, for any $t \in \{1, 2, \ldots, n\} \setminus \{j\}$, which contradicts that the j-th component does not satisfy the $*$-Property. Let us consider $\widetilde{b} = (b_1, b_2, \ldots, b_n) \in \ell_\infty^n$, where $b_j = 1$ and $b_k = 0$, for all $k \neq j$. Clearly, $\widetilde{b} \notin \mathbb{Y}$. Since $J(y)$ is a face of $B_{(\ell_\infty^n)^*}$ and there does not exist any $y \in \mathbb{Y}$ such that $J(y) = \{f_j\}$, we conclude that for any $y \in \mathbb{Y}$, there exists an $f \in Ext(B_{(\ell_\infty^n)^*}) \setminus \{\pm f_j\}$ such that $f \in J(y)$. As $f(\widetilde{b}) = 0$, for any $f \in Ext(B_{\ell_\infty^n}{}^*) \setminus \{\pm f_j\}$, by using Theorem 1.12 we obtain that 0 is a best coapproximation to $\widetilde{b}$ out of $\mathbb{Y}$. This contradicts that $\mathbb{Y}$ is an anti-coproximinal subspace of ℓ_∞^n .

To obtain (iii), we now need to show that $|P_i^+ \cup P_i^-| = 1$, for all $1 \leq i \leq n$. Clearly, $i \in P_i^+ \cup P_i^-$. Suppose on the contrary that $l \in P_i^+ \cup P_i^-$, for some $l \in \{1, 2, \ldots, n\} \setminus \{i\}$. Consider the element $\widetilde{b} = (b_1, b_2, \ldots, b_n) \in \ell_\infty^n$, where $b_l = 1$ and $b_k = 0$, for all $k \neq l$. Clearly, $\widetilde{b} \notin \mathbb{Y}$ and $f_k(\widetilde{b}) = 0$, for any $k \in \{1, 2, \ldots, n\} \setminus \{l\}$. As, $l \in P_i^+ \cup P_i^-$, for any $y \in \mathbb{Y}$, $|f_l(y)| = |f_i(y)|$. Therefore, if $f_l \in J(y)$, for some $y \in \mathbb{Y}$, then either $f_i \in J(y)$ or $-f_i \in J(y)$. We conclude that for any $y \in \mathbb{Y}$, there exists $f \in Ext(B_{\ell_\infty^n}{}^*) \setminus \{\pm f_l\}$ such that $f \in J(y)$. As, $f(\widetilde{b}) = 0$, for any $f \in Ext(B_{\ell_\infty^n}{}^*) \setminus \{\pm f_l\}$, by using Theorem 1.12 we obtain that 0 is a best coapproximation to $\widetilde{b}$ out of $\mathbb{Y}$. This again contradicts our hypothesis that $\mathbb{Y}$ is an anti-coproximinal subspace of ℓ_∞^n. This completes the proof.

Next we prove that (ii) $\implies$ (i). Suppose on the contrary that $\mathbb{Y}$ is not an anti-coproximinal in ℓ_∞^n. This implies there exists $x \in \ell_\infty^n \setminus \mathbb{Y}$ such that $\mathbb{Y} \perp_B x$. Following 1.6, we note that for each $y \in \mathbb{Y}$ there exists an $f \in J(y)$ such that $f(x) = 0$. Since each component of $\mathcal{A}$ satisfying the $*$-Property and $|P_i^+ \cup P_i^-| = 1$, it follows from Lemma 4.2 that for every $f \in Ext(B_{(\ell_\infty^n)^*})$ there exists a $\widetilde{y} \in Sm(\mathbb{X}) \cap S_{\mathbb{Y}}$ such that $J(\widetilde{y}) = \{f\}$. Therefore, $f(x) = 0$, for any $f \in Ext(B_{(\ell_\infty^n)^*})$. This implies that $|f(\widetilde{b} - y_0)| < \|\widetilde{b} - y_0\|$, for any $f \in Ext(B_{(\ell_\infty^n)^*})$. This is a contradiction and hence the theorem. □

The following example illustrates the computational effectiveness of Theorem 4.7 in verifying the anti-coproximinality of a given subspace of ℓ_∞^n.

Example 4.8. *Let* $\widetilde{a}_1 = (-4, 2, 3, 1, 3), \widetilde{a}_2 = (1, -5, 4, 2, -3), \widetilde{a}_3 = (1, 3, -7, 4, 6) \in \ell_\infty^5$ *and let* $\mathbb{Y} = span\{\widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3\}$. *The* 1*st,* 2*nd,* 3*rd,* 4*th and the* 5*th components are* $(-4, 1, 1)$, $(2, -5, 3)$, $(3, 4, -7)$, $(1, 2, 4)$ *and* $(3, -3, 6)$, *respectively. It is immediate that* $|P_i^+ \cup P_i^-| = 1$, *for all* $1 \leq$

$i \leq 5$. It is straightforward to verify that every component satisfies the $*$-Property. Therefore, following Theorem 4.7, $\mathbb{Y}$ is an anti-coproximinal subspace of ℓ_∞^5.

In light of the above example, we make the following remark that emphasizes the importance of the mother space in deciding whether a given subspace is strongly anti-coproximinal or not.

Remark 4.9. *Let $\mathbb{X} = \ell_\infty^6$ and let $\{e_1, e_2, \ldots, e_6\}$ be the standard ordered basis of $\mathbb{X}$. Suppose that $\mathbb{Z} = span\{e_1, e_2, \ldots, e_5\}$ is a subspace of $\mathbb{X}$. Let us now define $\psi : \ell_\infty^5 \to \mathbb{Z}$, as $\psi(x_1, x_2, \ldots, x_5) = (x_1, x_2, \ldots, x_5, 0)$, for any $(x_1, x_2, \ldots, x_5) \in \ell_\infty^5$. Clearly, ψ is an isometric isomorphism. Let $\mathbb{Y}$ be the same subspace given in Example 4.8. As $\mathbb{Y}$ is anti-coproximinal in ℓ_∞^5, clearly we observe that $\psi(\mathbb{Y})$ is also anti-coproximinal in $\mathbb{Z}$. However, $\psi(\mathbb{Y})$ is not an anti-coproximinal subspace of $\mathbb{X}$, as $\psi(\widetilde{a}_1) = (-4, 2, 3, 1, 3, 0), \psi(\widetilde{a}_2) = (1, -5, 4, 2, -3, 0), \psi(\widetilde{a}_3) = (1, 3, -7, 4, 6, 0)$ and the 6th component does not satisfy the $*$-Property (see Theorem 4.18).*

In Chapter 3, the notions 'the zero set($\mathcal{Z}_\mathcal{A}$)' and 'the minimal norming set($\mathcal{N}$)' have been introduced to study the best coapproximation problem in ℓ_1^n. Using these notions, we study the anti-coproximinal subspaces and the strongly anti-coproximinal subspaces in ℓ_1^n. Let us now recall the definitions of the zero set and the minimal norming set.

Definition 4.3. *[53] Let $\mathcal{A} = \{\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_m\}$ be a set of linearly independent elements in ℓ_1^n, where $1 \leq m \leq n$ and $\widetilde{a}_k = (a_1^k, a_2^k, \ldots, a_n^k)$, for each $1 \leq k \leq m$. The* zero set *$\mathcal{Z}_\mathcal{A}$ of $\mathcal{A}$ is defined as*

$$\mathcal{Z}_\mathcal{A} = \Big\{ i \in \{1, 2, \ldots, n\} : \big(a_i^1, a_i^2, \ldots, a_i^m\big) = (0, 0, \ldots, 0) \Big\}.$$

Definition 4.4. *[53] A set S in a Banach space $\mathbb{X}$ is said to be symmetric if $x \in S$ implies $-x \in S$. Let $\mathbb{Y}$ be a subspace of ℓ_1^n. A symmetric set $\mathcal{N}$ is said to be a norming set of $\mathbb{Y}$ if $\big(M_g \cap Ext(B_{\ell_\infty^n})\big) \cap \mathcal{N} \neq \emptyset$, for each $g \in \psi(\mathbb{Y})$, where ψ is the canonical isometric isomorphism between ℓ_1^n and $(\ell_\infty^n)^*$. A norming set $\mathcal{N}$ is said to be a minimal norming set of $\mathbb{Y}$ if for any norming set $\mathcal{M}$ of $\mathbb{Y}$, $\mathcal{M} \subset \mathcal{N}$ implies that $\mathcal{M} = \mathcal{N}$.*

In the following theorem we completely characterize the anti-coproximinal subspaces in ℓ_1^n in terms of the minimal norming set. As we will observe, this characterization turns out to be particularly helpful for explicitly describing the anti-coproximinal subspaces in ℓ_1^n.

Theorem 4.10. *Let $\mathbb{Y}$ be a subspace of ℓ_1^n and let $\mathcal{A} = \{\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_m\}$ be a basis of $\mathbb{Y}$, where $1 < m < n$ and $\widetilde{a}_k = (a_1^k, a_2^k, \ldots, a_n^k)$, for each $1 \leq k \leq m$. Then the following statements hold true:*

(i) If $\mathcal{Z}_\mathcal{A} \neq \emptyset$, then $\mathbb{Y}$ is not an anti-coproximinal subspace of ℓ_1^n.

(ii) If $\mathcal{Z}_{\mathcal{A}} = \emptyset$, then $\mathbb{Y}$ is an anti-coproximinal subspace of ℓ_1^n if and only if $dim(span\ \mathcal{N}) = n$, where $\mathcal{N}$ is the minimal norming set.

Proof. (i) Suppose that $j \in \mathcal{Z}_{\mathcal{A}}$, i.e., $a_j^k = 0$, for all $k \in \{1, 2, \ldots, n\}$. Let $\widetilde{b} = (b_1, b_2, \ldots, b_n) \in \ell_1^n$, where $b_j = 1$ and $b_i = 0$, for any $i \in \{1, 2, \ldots, n\} \setminus \{j\}$. Clearly, $\widetilde{b} \notin \mathbb{Y}$. For any $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{R}$, it is easy to see that

$$\|\widetilde{b} - \sum_{i=1}^{m} \beta_i \widetilde{a}_i\| = \|\sum_{i=1}^{m} \beta_i \widetilde{a}_i\| + 1 > \|\sum_{i=1}^{m} \beta_i \widetilde{a}_i - 0\|.$$

Therefore, 0 is a best coapproximation to $\widetilde{b}$ out of $\mathbb{Y}$ and consequently $\mathbb{Y}$ is not an anti-coproximinal subspace of ℓ_1^n.

(ii) Since $\mathcal{Z}_{\mathcal{A}} = \emptyset$, it follows from [53, Th 2.2] that the minimal norming set $\mathcal{N}$ is unique. Suppose that $\mathcal{N} = \{\pm\widetilde{x}_1, \pm\widetilde{x}_2, \ldots, \pm\widetilde{x}_q\}$, where $\widetilde{x}_k = (x_1^k, x_2^k, \ldots, x_n^k)$. Without loss of generality we assume that $\{\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_q\}$ is linearly independent.
Let us first prove the necessary part. Suppose on the contrary that $q < n$. It is straightforward to see that there exists a non zero element $\widetilde{b} = (b_1, b_2, \ldots, b_n) \in \ell_1^n$ such that $\sum_{i=1}^{n} b_i x_i^p = 0$, for any $1 \leq p \leq q$. Observe that $\widetilde{b} \notin \mathbb{Y}$. Otherwise, there exists an $\widetilde{x}_k \in \mathcal{N}$ such that $\widetilde{x}_k \in M_{\psi(\widetilde{b})}$, for some $1 \leq k \leq q$, i.e.,

$$\psi(\widetilde{b})(\widetilde{x}_k) = \sum_{i=1}^{n} b_i x_i^k \neq 0,$$

where $\psi : \ell_1^n \to (\ell_\infty^n)^*$ is the canonical isometric isomorphism. Therefore, for $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$, the following system of linear equations holds true:

$$\alpha_1 \sum_{i=1}^{n} a_i^1 x_i^p + \alpha_2 \sum_{i=1}^{n} a_i^2 x_i^p + \ldots + \alpha_m \sum_{i=1}^{n} a_i^m x_i^p = 0 = \sum_{i=1}^{n} b_i x_i^p,$$

where $p \in \{1, 2, \ldots, q\}$. Now applying Theorem [53, Th. 2.4], it is easy to observe that 0 is a best coapproximation to $\widetilde{b}$ out of $\mathbb{Y}$. This contradicts the fact that $\mathbb{Y}$ is an anti-coproximinal subspace of ℓ_1^n. Thus we obtain $q = n$. This proves the necessary part.

We now prove the sufficient part. Let $\mathcal{N}$ be the minimal norming set and let $|\mathcal{N}| = q$. Suppose on the contrary that $\mathbb{Y}$ is not an anti-coproximinal subspace of ℓ_1^n. Then there exists $\widetilde{b} = (b_1, b_2, \ldots, b_n) \in \ell_1^n \setminus \mathbb{Y}$ such that y_0 is a best coapproximation to $\widetilde{b}$ out of $\mathbb{Y}$. It follows that 0 is a best coapproximation to $\widetilde{b} - y_0$ out of $\mathbb{Y}$. Assume that $\widetilde{b} - y_0 = \widetilde{d} = (d_1, d_2, \ldots, d_n)$. Clearly, $\widetilde{d} \in \ell_1^n \setminus \mathbb{Y}$. Applying Theorem [53, Th. 2.4], we obtain that $\sum_{i=1}^{n} d_i x_i^p = 0$, for each $1 \leq p \leq q$. Since $dim(span\ \mathcal{N}) = dim(span\{(x_1^p, x_2^p, \ldots, x_n^p) : 1 \leq p \leq q\}) = n$, it follows that

$\widetilde{d} = (d_1, d_2, \dots, d_n) = 0$. This contradicts that $\widetilde{d} \in \ell_1^n \setminus \mathbb{Y}$. This completes the proof. □

Next we give an example of an anti-coproximinal subspace of ℓ_1^3 by applying Theorem 4.10.

Example 4.11. *Let $\mathcal{A} = \{(0,1,1), (-1,0,1)\} \subset \ell_1^3$ and let $\mathbb{Y} = span\ \mathcal{A}$. It is easy to observe that $\mathcal{Z}_{\mathcal{A}} = \emptyset$. Then following the same technique as given in the proof of [53, Th. 2.2], we obtain that the minimal norming set $\mathcal{N}$ of $\mathbb{Y}$ as $\{\pm(1,1,1), \pm(-1,1,1), \pm(-1,-1,1)\}$. Thus we get $dim(span\ \mathcal{N}) = 3$. Applying Theorem 4.10(ii) we conclude that $\mathbb{Y}$ is an anti-coproximinal subspace of ℓ_1^n.*

4.3 Strongly anti-coproximinal subspace

4.3.1 Definition and basic properties

Observing the connection between the best coapproximation and Birkhoff-James orthogonality, we can also define an approximate version of the notion of best coapproximation with the help of approximate Birkhoff-James orthogonality. Let us now introduce the following definition:

Definition 4.5. *Let $\mathbb{Y}$ be a subspace of Banach space $\mathbb{X}$. Let $\epsilon \in [0,1)$. Given any $x \in \mathbb{X}$, we say that $y_0 \in \mathbb{Y}$ is an ϵ-best coapproximation to x out of $\mathbb{Y}$ if $\mathbb{Y} \perp_B^\epsilon x - y_0$, i.e., $y \perp_B^\epsilon (x - y_0)$, for all $y \in \mathbb{Y}$.*

We next obtain a characterization of the ϵ-best coapproximation, by applying Theorem 1.9.

Proposition 4.3. *Let $\mathbb{Y}$ be a subspace of a Banach space $\mathbb{X}$. Then the following statements are equivalent:*

(i) y_0 is an ϵ-best coapproximation to x out of $\mathbb{Y}$

(ii) for any $y \in \mathbb{Y}$, $\|x - y\| \geq \|y_0 - y\| - \epsilon\|x - y_0\|$

(iii) for any $y \in \mathbb{Y}$, there exists $f_y \in J(y)$ such that $|f_y(x - y_0)| \leq \epsilon\|x - y_0\|$.

Proof. $(i) \Rightarrow (ii)$: Let y_0 be an ϵ-best coapproximation to x out of $\mathbb{Y}$. Therefore, $\mathbb{Y} \perp_B^\epsilon (x - y_0)$, i.e., $y \perp_B^\epsilon (x - y_0)$, for any $y \in \mathbb{Y}$. Thus for any $y \in \mathbb{Y}$, $\|y + \lambda(x - y_0)\| \geq \|y\| - \epsilon\|\lambda(x - y_0)\|$, for all $\lambda \in \mathbb{R}$. Putting $\lambda = 1$, we get $\|y + (x - y_0)\| \geq \|y\| - \epsilon\|(x - y_0)\|$, from which the desired inequality follows easily.
$(ii) \Rightarrow (iii)$: For any $y \in \mathbb{Y}$ and for any nonzero $\lambda \in \mathbb{R}$,

$$\|y + \lambda(x - y_0)\| \quad = \quad |\lambda|\|x - (y_0 - \frac{1}{\lambda}y)\|$$

$$\begin{aligned} &\geq |\lambda|\{\|y_0 - (y_0 - \frac{1}{\lambda}y)\| - \epsilon\|x - y_0\|\} \\ &= |\lambda|(\|\frac{1}{\lambda}y\| - \epsilon\|x - y_0\|) \\ &= \|y\| - \epsilon\|\lambda(x - y_0)\|. \end{aligned}$$

For $\lambda = 0$, the above inequality holds trivially. This implies $y \perp_B^{\epsilon} (x - y_0)$, for any $y \in \mathbb{Y}$ and so the result follows from Theorem 1.9.

$(iii) \Rightarrow (i)$: Following Theorem 1.9, we get $y \perp_B^{\epsilon} x - y_0$, for all $y \in \mathbb{Y}$. Thus we obtain $\mathbb{Y} \perp_B^{\epsilon} x - y_0$. □

The primary purpose of our study is to investigate the least favorable scenario that can arise in studying the best coapproximation problem. Accordingly, we introduce the following type of subspaces of a Banach space from the perspective of ϵ-best coapproximation, a much stronger version than anti-coproximinal subspaces.

Definition 4.6. *Let $\mathbb{Y}$ be a subspace of Banach space $\mathbb{X}$. Then $\mathbb{Y}$ is said to be an anti-coproximinal subspace of $\mathbb{X}$ if for any given $x \in \mathbb{X} \setminus \mathbb{Y}$, there does not exist any best coapproximation to x out of $\mathbb{Y}$.*

Clearly, a strongly anti-coproximinal subspace is an anti-coproximinal subspace. Similar to the anti-coproximinality notions, the dense subspaces are all strongly anti-coproximinal.

4.3.2 Strongly anti-coproximinal subspaces in general Banach space

Our next goal is to separately present a necessary condition and a sufficient condition for strongly anti-coproximinal subspaces of a Banach space. First we give the sufficient condition.

Theorem 4.12. *Let $\mathbb{Y}$ be a subspace of a Banach space $\mathbb{X}$. Then $\mathbb{Y}$ is a strongly anti-coproximinal subspace of $\mathbb{X}$ if for each $x \in \mathbb{X} \setminus \mathbb{Y}$, there exists a $y \in \mathbb{Y}$ such that $J(y) \subseteq J(x) \cup J(-x)$.*

Proof. Suppose on the contrary that $\mathbb{Y}$ is not a strongly anti-coproximinal subspace of $\mathbb{X}$. Therefore, there exists an $x \in \mathbb{X} \setminus \mathbb{Y}$ such that $y_1 \in \mathbb{Y}$ is an ϵ-best coapproximation to x out of $\mathbb{Y}$, for some $\epsilon \in [0, 1)$. Applying Proposition 4.3, we obtain that for each $y \in \mathbb{Y}$, there exists an $f_y \in J(y)$ such that $|f_y(x - y_1)| \leq \epsilon\|x - y_1\| < \|x - y_1\|$. Therefore, for each $y \in \mathbb{Y}$, there exists an $f_y \in J(y)$, such that $f_y \notin J(x - y_1) \cup J(-(x - y_1))$. This contradicts the hypothesis of the theorem, thereby finishing the proof. □

Let us now present a necessary condition for strongly anti-coproximinal subspaces of a Banach space under some additional nice conditions.

Theorem 4.13. *Let $\mathbb{X}$ be a reflexive Banach space and let $\mathbb{X}^*$ satisfies the Kadets-Klee Property. Let $\mathbb{Y}$ be a closed subspace of $\mathbb{X}$. If $\mathbb{Y}$ is a strongly anti-coproximinal subspace of $\mathbb{X}$ then for each $x \in \mathbb{X}$, there exists an element $y \in \mathbb{Y}$ such that $J(y) \cap J(x) \neq \emptyset$.*

Proof. Note that whenever $x \in \mathbb{Y}$, we have nothing to prove. Take $x \in \mathbb{X} \setminus \mathbb{Y}$. Since $\mathbb{Y}$ is a strongly anti-coproximinal subspace of $\mathbb{X}$, it follows that for any $\epsilon \in [0, 1)$, $\mathbb{Y} \not\perp_B^{\epsilon} x$. Let us take a sequence $\{\epsilon_n\}_{n \in \mathbb{N}} \subset [0, 1) \to 1$, as $n \to \infty$. Suppose that for each $n \in \mathbb{N}$, there exists $y_n \in S_{\mathbb{Y}}$ such that $y_n \not\perp_B^{\epsilon_n} x$. From Theorem 1.9, we obtain that for any $f_n \in J(y_n)$, $|f_n(x)| > \epsilon_n \|x\|$. Since $\mathbb{X}$ is reflexive, it follows that $\mathbb{X}^*$ is reflexive and therefore without loss of generality we may and do assume that f_n is weakly convergent to f, for some $f \in B_{\mathbb{X}^*}$. So, $f_n(x) \to f(x)$. Taking limit on the both sides of the above inequality, we obtain $|f(x)| \geq \|x\|$. Since $f \in B_{\mathbb{X}^*}$, $|f(x)| = \|x\|$, and therefore, $\|f\| = 1$. Thus either $f \in J(x)$ or $-f \in J(x)$. Also $\|f_n\| \to \|f\|$ as $n \to \infty$. Since $\mathbb{X}^*$ satisfies the Kadets-Klee Property, it follows that $f_n \to f$ as $n \to \infty$. As $\mathbb{X}$ is reflexive and $\mathbb{Y}$ is a closed subspace of $\mathbb{X}$, then $\mathbb{Y}$ is also reflexive, and therefore $B_{\mathbb{Y}}$ is weakly compact. So, y_n weakly converges to y, for some $y \in B_{\mathbb{Y}}$. Now as $f_n \to f$ and $y_n \xrightarrow{w} y$, it is straightforward to see that $f(y) = 1$. Therefore, $f \in J(y)$ and consequently, either $J(y) \cap J(x) \neq \emptyset$ or $J(-y) \cap J(x) \neq \emptyset$. This completes the theorem. □

Remark 4.14. *Observe that the above condition is only necessary but not sufficient, see Example 4.19.*

Applying Theorem 4.13, it is possible to give examples of Banach spaces which do not contain any strongly anti-coproximinal closed subspaces.

Theorem 4.15. *Let $\mathbb{X}$ be a reflexive Banach space and let $\mathbb{X}^*$ satisfies the Kadets-Klee Property. Suppose that $\mathbb{Y}$ is a closed subspace of $\mathbb{X}$ such that there exists a rotund point in $S_{\mathbb{X}} \setminus S_{\mathbb{Y}}$. Then $\mathbb{Y}$ is not a strongly anti-coproximinal subspace of $\mathbb{X}$. In particular, every reflexive strictly convex Banach space, whose dual satisfies the Kadets-Klee Property, does not contain any strongly anti-coproximinal closed subspaces.*

Proof. Suppose that $x \in S_{\mathbb{X}} \setminus S_{\mathbb{Y}}$ is a rotund point. It is straightforward to see that for any $y \in \mathbb{Y}$, $J(x) \cap J(y) = \emptyset$. Indeed, if possible let $y_0 \in S_{\mathbb{Y}}$ be such that $f \in J(x) \cap J(y_0)$. This implies that $f(\frac{x+y_0}{2}) = 1 \implies \|\frac{x+y_0}{2}\| = 1$. Since x is rotund, it follows that $x = y_0$, a contradiction. Now applying Theorem 4.13, it can be concluded that $\mathbb{Y}$ is not a strongly anti-coproximinal subspace of $\mathbb{X}$. This completes the proof of the first part. The second part follows trivially from the first part. □

Our next result shows that the condition of strict convexity in the previous theorem can be replaced by the condition of smoothness.

Theorem 4.16. *Let $\mathbb{X}$ be a reflexive smooth Banach space and let $\mathbb{X}^*$ satisfies the Kadets-Klee Property. Suppose that $\mathbb{Y}$ is a closed subspace of $\mathbb{X}$. Then $\mathbb{Y}$ is not a strongly anti-coproximinal subspace of $\mathbb{X}$.*

Proof. Suppose that $\mathbb{Y}$ is a closed subspace of $\mathbb{X}$. Let us consider $g \in \mathbb{Y}^\perp$ such that $\|g\| = 1$. Since $\mathbb{X}$ is reflexive, there exists a $z \in S_{\mathbb{X}}$ such that $|g(z)| = 1$. Therefore, $J(z) = \{g\}$ or $J(z) = \{-g\}$, as $\mathbb{X}$ is smooth. We claim that for any $y \in \mathbb{Y}$, $J(y) \cap J(z) = \emptyset$. Otherwise, take a nonzero element $y_1 \in \mathbb{Y}$ such that $J(y_1) \cap J(z) \neq \emptyset$. Since $\mathbb{X}$ is smooth, it follows that either $J(y_1) = \{g\}$ or $J(y_1) = \{-g\}$. Then $|g(y_1)| = \|y_1\|$, which contradicts the fact that $g \in \mathbb{Y}^\perp$. Therefore, applying Theorem 4.13, we conclude that $\mathbb{Y}$ is not a strongly anti-coproximinal subspace of $\mathbb{X}$. This completes the theorem. □

We end this section with the following result, which is an immediate corollary of Theorem 4.13.

Corollary 4.2. *Let $\mathbb{Y}$ be a closed subspace of a Banach space $\mathbb{X}$. Suppose that $\mathbb{X}$ satisfies either of the following properties:*

(i) *$\mathbb{X}$ is a finite-dimensional smooth Banach space*

(ii) *$\mathbb{X}$ is a finite-dimensional strictly convex Banach space*

(iii) *$\mathbb{X}$ is a uniformly smooth Banach space.*

Then $\mathbb{Y}$ is not a strongly anti-coproximinal subspace of $\mathbb{X}$.

4.3.3 Strongly anti-coproximinal subspaces in polyhedral Banach spaces

We next characterize the strongly anti-coproximinal subspaces in finite-dimensional polyhedral Banach spaces.

Theorem 4.17. *Let $\mathbb{Y}$ be a subspace of a finite-dimensional polyhedral Banach space $\mathbb{X}$. Then the following statements are equivalent:*

(i) *$\mathbb{Y}$ is a strongly anti-coproximinal subspace of $\mathbb{X}$*

(ii) *$\mathbb{Y}$ intersects the interior of every facet of $B_{\mathbb{X}}$*

(iii) *$\mathcal{J}_{\mathbb{Y}} = Ext(B_{\mathbb{X}^*})$.*

Proof. Suppose that $\pm F_1, \pm F_2, \ldots, \pm F_r$ are the facets of $B_{\mathbb{X}}$. Following from [48, Lemma 2.1], assume that $\pm f_1, \pm f_2, \ldots, \pm f_r$ are the corresponding extreme points of $B_{\mathbb{X}^*}$, respectively. Clearly, $Ext(B_{\mathbb{X}^*}) = \{\pm f_1, \pm f_2, \ldots, \pm f_r\}$. We complete the proof in the following three steps:
(i) $\implies$ (ii): Suppose on the contrary that $\mathbb{Y}$ does not intersect the interior of the facet F_i, for some $i \in \{1, 2, \ldots, r\}$. Take $x \in int(F_i)$. Clearly, $x \notin \mathbb{Y}$. Moreover, x is smooth and $J(x) = \{f_i\}$. Since $\mathbb{Y}$ does not intersect the interior of F_i, it is easy to observe that for any $y \in \mathbb{Y}$, $J(y) \cap (Ext(B_{\mathbb{X}^*}) \setminus \{\pm f_i\}) \neq \emptyset$, otherwise, $J(y) = \{f_i\}$. Take $\epsilon_0 = \max\{|f(x)| : f \in Ext(B_{\mathbb{X}^*}) \setminus \{\pm f_i\}\}$. As $J(x) = \{f_i\}$, it is clear that $\epsilon_0 < 1$. Since for any $y \in \mathbb{Y}$, $J(y) \cap (Ext(B_{\mathbb{X}^*}) \setminus \{\pm f_i\}) \neq \emptyset$, it is easy to see that for any $y \in \mathbb{Y}$ there exists an $f \in J(y)$ such that $|f(x)| \leq \epsilon_0$. Following Theorem 1.9, $y \perp_B^{\epsilon_0} x$, for any $y \in \mathbb{Y}$. In other words, 0 is an ϵ_0-best coapproximation to x out of $\mathbb{Y}$. This contradicts that $\mathbb{Y}$ is a strongly anti-coproximinal subspace of $\mathbb{X}$.
(ii) $\implies$ (iii): Suppose that $y_i \in int(F_i) \cap \mathbb{Y}$, for each $i \in \{1, 2, \ldots, r\}$. Clearly, y_i is smooth and $J(y_i) = \{f_i\}$. Therefore, $f_i \in \mathcal{J}_{\mathbb{Y}}$, for each $i \in \{1, 2, \ldots, r\}$, this implies that $Ext(B_{\mathbb{X}^*}) \subset \mathcal{J}_{\mathbb{Y}}$. So, $Ext(B_{\mathbb{X}^*}) = \mathcal{J}_{\mathbb{Y}}$.
(iii) $\implies$ (i): Let $x \in \mathbb{X}$. Without loss of generality we assume that $x \in F_i$, for some $i \in \{1, 2, \ldots, r\}$. Clearly, $f_i \in J(x)$. Since $\mathcal{J}_{\mathbb{Y}} = Ext(B_{\mathbb{X}^*})$, there exists a $y \in Sm(\mathbb{X}) \cap \mathbb{Y}$ such that $J(y) = \{f_i\}$. Therefore, $J(y) = \{f_i\} \subseteq J(x)$. By applying Theorem 4.12, we obtain that $\mathbb{Y}$ is a strongly anti-coproximinal subspace of $\mathbb{X}$. □

The above characterization for strongly anti-coproximinal subspaces is geometric in nature, it also depicted the extreme nature of these type of subspaces. Although we obtain a complete characterization, a computationally effective characterization is needed for the space ℓ_∞^n and ℓ_1^n.

Theorem 4.18. *Let $\mathcal{A} = \{\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_m\}$ be a set of linearly independent elements in ℓ_∞^n, where $1 < m < n$ and $\widetilde{a}_k = (a_1^k, a_2^k, \ldots, a_n^k)$, for each $1 \leq k \leq m$. Let $\mathbb{Y} = span\ \mathcal{A}$. Then the following statements are equivalent:*

(i) $\mathbb{Y}$ is a strongly anti-coproximinal subspace of ℓ_∞^n.

(ii) $\mathbb{Y}$ is an anti-coproximinal subspace of ℓ_∞^n.

(iii) Every component of $\mathcal{A}$ satisfies the $*$-Property and $|P_i^+ \cup P_i^-| = 1$, for all $1 \leq i \leq n$.

Proof. We begin the proof by noting that (i) $\implies$ (ii) follows trivially and (ii) $\implies$ (iii) follows from Theorem 4.7.

We only need to prove (iii) $\implies$ (i). Suppose on the contrary that y_0 is an ϵ-best coapproximation to $\widetilde{b} \in \ell_\infty^n \setminus \mathbb{Y}$ out of $\mathbb{Y}$. Following Proposition 4.3(iii), we note that for each $y \in \mathbb{Y}$ there exists an $f \in J(y)$ such that $|f(\widetilde{b} - y_0)| \leq \epsilon \|\widetilde{b} - y_0\|$, for some $\epsilon \in [0, 1)$. Since each component of $\mathcal{A}$ satisfying the $*$-Property and $|P_i^+ \cup P_i^-| = 1$, it follows from Lemma 4.2 that for every $f \in Ext(B_{(\ell_\infty^n)^*})$ there exists a $\widetilde{y} \in Sm(\mathbb{X}) \cap S_{\mathbb{Y}}$ such that $J(\widetilde{y}) = \{f\}$. Therefore, $|f(\widetilde{b} - y_0)| \leq \epsilon \|\widetilde{b} - y_0\|$, for any $f \in Ext(B_{(\ell_\infty^n)^*})$. This implies that $|f(\widetilde{b} - y_0)| < \|\widetilde{b} - y_0\|$, for any $f \in Ext(B_{(\ell_\infty^n)^*})$. Therefore, it is easy to observe that

$$\|\widetilde{b} - y_0\| = \sup_{\|f\| \leq 1} \big\{|f(\widetilde{b} - y_0)|\big\} = \max_{f \in Ext(B_{(\ell_\infty^n)^*})} \big\{|f(\widetilde{b} - y_0)|\big\} < \|\widetilde{b} - y_0\|,$$

a contradiction. Hence (iii) $\implies$ (i). □

As promised earlier, here we present an example of a subspace which is not strongly anti-coproximinal but satisfies the necessary condition of Theorem 4.13.

Example 4.19. *Let $\widetilde{a}_1 = (1, 1, 2), \widetilde{a}_2 = (2, 2, 1) \in \ell_\infty^3$ and let $\mathbb{Y} = span\{\widetilde{a}_1, \widetilde{a}_2\}$. The 1st, 2nd, 3rd components are $(1, 2), (1, 2), (2, 1)$, respectively. It is immediate that $|P_1^+ \cup P_1^-| = 2$. Therefore, following Theorem 4.18, $\mathbb{Y}$ is not a strongly anti-coproximinal subspace of ℓ_∞^3. However, by a straightforward observation we can verify that for any $x \in \ell_\infty^3$, there exists a $y \in \mathbb{Y}$ such that $J(y) \cap J(x) \neq \emptyset$.*

Using Theorem 4.17 we show that there does not exist any strongly anti-coproximinal subspace in ℓ_1^n. Before proving the result, we note the following lemma.

Lemma 4.3. *Let $\mathbb{Y}$ be a strongly anti-coproximinal subspace of ℓ_1^n. Then $\phi(\mathcal{J}_{\mathbb{Y}}) = \mathcal{N}$, where $\mathcal{N}$ is the minimal norming set and ϕ is the canonical isometric isomorphism between $(\ell_1^n)^*$ and ℓ_∞^n.*

Proof. Since $\mathbb{Y}$ is a strongly anti-coproximinal subspace of ℓ_1^n, it follows from Theorem 4.10 that $\mathcal{Z}_{\mathcal{A}} = \emptyset$. Suppose that $\psi : \ell_1^n \to (\ell_\infty^n)^*$ is the canonical isometric isomorphism. Let us first assume that $f \in \mathcal{J}_{\mathbb{Y}}$. This implies that there exists a $y \in Sm(\ell_1^n) \cap S_{\mathbb{Y}}$ such that $f(y) = 1$. Now observe that $\psi(y)(\phi(f)) = f(y) = 1$. As $\psi(y)$ is a smooth point in $(\ell_\infty^n)^*$, we have $M_{\psi(y)} = \{\pm\phi(f)\}$. Therefore, $\phi(\mathcal{J}_{\mathbb{Y}}) \subset \mathcal{N}$. On the other hand, since $\mathbb{Y}$ is a strongly anti-coproximinal subspace of ℓ_1^n, it follows from Theorem 4.17 that

$$\mathcal{N} \subset Ext(B_{\ell_\infty^n}) = \phi(Ext(B_{(\ell_1^n)^*})) = \phi(\mathcal{J}_{\mathbb{Y}}).$$

This proves our lemma. □

Let us now present the desired result.

Theorem 4.20. *There is no strongly anti-coproximinal subspace in ℓ_1^n.*

Proof. Suppose on the contrary we assume that $\mathbb{Y}$ is a strongly anti-coproximinal subspace in ℓ_1^n. Clearly, $\mathbb{Y}$ is a proper subspace of ℓ_1^n. Let $\mathcal{A} = \{\widetilde{a_1}, \widetilde{a_2}, \ldots, \widetilde{a_m}\}$ be a basis of $\mathbb{Y}$, where $\widetilde{a_k} = (a_1^k, a_2^k, \ldots, a_n^k)$, for all $k \in \{1, 2, \ldots, m\}$, where $1 < m < n$. It follows from Theorem 4.10 that $\mathcal{Z}_{\mathcal{A}} = \emptyset$. Also from Theorem 4.17, we have $|\mathcal{J}_{\mathbb{Y}}| = |Ext(B_{(\ell_1^n)^*})| = 2^n$. By virtue of Lemma 4.3, we note that the cardinality of the minimal norming set of $\mathbb{Y}$ is equal to the cardinality of the set $\mathcal{J}_{\mathbb{Y}}$. Let us now consider the following hyperspaces in $\mathbb{R}^m$

$$H_i = \left\{(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{R}^m : \sum_{k=1}^{m} \beta_k a_i^k = 0\right\},$$

where $i \in \{1, 2, \ldots, n\}$. Now it is easy to observe from [53, Th. 2.2] that there is an one-one correspondence between the minimal norming set of $\mathbb{Y}$ and the set of regions $\mathbb{R}^m$ formed by H_i's for all $i \in \{1, 2, \ldots, n\}$. Applying [18, Th. 1], we obtain that these hyperspaces divide $\mathbb{R}^m$ into at most $2[\binom{n-1}{0} + \binom{n-1}{1} + \ldots + \binom{n-1}{m-1}]$ regions. Therefore the cardinality of the minimal norming set of $\mathbb{Y}$ is at most $2[\binom{n-1}{0} + \binom{n-1}{1} + \ldots + \binom{n-1}{m-1}]$. Since $m < n$, it is immediate that

$$2\left[\binom{n-1}{0} + \binom{n-1}{1} + \ldots + \binom{n-1}{m-1}\right] < 2^n.$$

This contradicts that the cardinality of the minimal norming set of $\mathbb{Y}$ is 2^n. This completes the proof of the theorem. □

4.4 Concluding remarks

We would like to end this chapter with the following remarks regarding the anti-coproximinal subspaces and the strongly anti-coproximinal subspaces.

Remark 4.21. *(i) We have already shown that there exists a strongly anti-coproximinal subspace in* $\ell_\infty^n (n \geq 3)$*, whereas there does not exist any strongly anti-coproximinal subspace in* $\ell_1^n (n \geq 3)$*. Using Theorem 4.17, it is possible to give a geometric interpretation of this phenomenon in a visually appealing manner. Indeed, we observe that there is a subspace in* ℓ_∞^n *which intersects the interior of each of the facet of* $B_{\ell_\infty^n}$*, whereas* ℓ_1^n *does not contain any such subspaces (see Theorem 4.20).*

(ii) We note that the anti-coproximinal subspaces are the least favorable subspaces from the perspective of best coapproximation. In this study we have observed that in general there may be many subspaces in a polyhedral Banach spaces which are anti-coproximinal. This further illustrates the non-triviality and the computational difficulty associated with the best coapproximation problem, even in finite-dimensional Banach spaces. We note from [19, Th. 4] that a

Banach space $\mathbb{X}$ *having three or more dimension is an inner product space if and only if for each hyperspace* H *of* $\mathbb{X}$*, there exists an* $x \in \mathbb{X}$ *such that* $H \perp_B x$*. In any Banach space, it is easy to verify that the anti-coproximinal hyperspaces are precisely those which are not orthogonal to any element of* $\mathbb{X}$*. Moreover, a strongly anti-coproximinal hyperspace* H *is precisely those for which there does not exist any element* $x \in \mathbb{X}$ *such that that* $H \perp_B^{\epsilon} x$*, for some* $\epsilon \in [0, 1)$*. From Theorem 4.20, for any hyperspace* H *of* ℓ_1^n*, we note that there exists an* $\epsilon \in [0, 1)$ *and an* $x \in \ell_1^n$ *such that* $H \perp_B^{\epsilon} x$*.*

(iii) It is known [26, 33] that given a subspace $\mathbb{Y}$ *of a Banach space* $\mathbb{X}$ *and an element* $x \notin \mathbb{Y}$*,* y_0 *is a best coapproximation to* x *out of* $\mathbb{Y}$ *if and only if there exists a norm one projection from* $span\{x, \mathbb{Y}\}$ *to* $\mathbb{Y}$*. It is clear that* $\mathbb{Y}$ is an anti-coproximinal subspace in $\mathbb{X}$ if and only if given any $x \notin \mathbb{Y}$, there does not exist any norm one projection from $span\{x, \mathbb{Y}\}$ to $\mathbb{Y}$. In other words, $\mathbb{Y}$ is an anti-coproximinal subspace in $\mathbb{X}$ if and only if for any subspace $\mathbb{Z}$ which properly contains $\mathbb{Y}$, there does not exist any norm one projection from $\mathbb{Z}$ to $\mathbb{Y}$.

CHAPTER 5

ANTI-COPROXIMINALITY IN THE SPACE OF ALL CONTINUOUS FUNCTION

5.1 Introduction

In Chapter 4, we introduced two classes of subspaces—anti-coproximinal and strongly anti-coproximinal—which serve as critical counterexamples in the context of the best coapproximation problem. These subspaces highlight situations where best coapproximations do not exist. We explored their structural properties and established necessary and sufficient conditions for their existence, particularly in smooth and strictly convex Banach spaces. Building on this foundation, the present chapter turns to the study of these subspaces in the setting of continuous function spaces. We begin by analyzing the space $C(K)$ of scalar-valued continuous functions on a compact Hausdorff space K. We then extend our investigation to the space of vector-valued continuous functions, where to examine the stability and structural behavior of

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these subspaces in more general contexts. It is time to mention the basic terminologies and the notations to be used throughout the chapter.

Let $\mathbb{X}, \mathbb{Y}$ denote Banach spaces over the field $\mathbb{K}$, either real or complex. For $\epsilon > 0$, let us set $\mathcal{D}(\epsilon) = \{z \in \mathbb{K} : |z| \leq \epsilon\}$ and let $\mathbb{T}$ denote the unit circle in the complex plane. We use the notations $B_{\mathbb{X}}$ and $S_{\mathbb{X}}$ to denote the unit ball and unit sphere of $\mathbb{X}$, respectively. The dual space of $\mathbb{X}$ is denoted by $\mathbb{X}^*$. For a non-zero $x \in \mathbb{X}$, $x^* \in S_{\mathbb{X}^*}$ is said to be a supporting functional at x if $x^*(x) = \|x\|$. The set of all supporting functionals at x is denoted by $J(x)$, i.e., $J(x) = \{x^* \in S_{\mathbb{X}^*} : x^*(x) = \|x\|\}$. A non-zero element $x \in \mathbb{X}$ is said to be smooth if $J(x)$ is a singleton. The collection of all smooth points in $\mathbb{X}$ is denoted by $Sm(\mathbb{X})$. A Banach space $\mathbb{X}$ is said to be smooth if $Sm(\mathbb{X}) = \mathbb{X} \setminus \{0\}$. The convex hull of a set S is denoted as $co(S)$. For a convex set C, an element $x \in C$ is said to be an extreme point if $x = (1-t)y + tz$, for some $t \in (0,1)$ and some $y, z \in C$ implies that $x = y = z$. The set of all extreme points of C is denoted by $Ext(C)$. A finite-dimensional real Banach space is said to be polyhedral if $B_{\mathbb{X}}$ is a polyhedron, or, equivalently, if $Ext(B_{\mathbb{X}})$ is finite. A convex set $F \subset S_{\mathbb{X}}$ is said to be a face of $B_{\mathbb{X}}$ if for any $y, z \in B_{\mathbb{X}}$, $\frac{1}{2}(y+z) \in F$ implies that $y, z \in F$. F is called a maximal face if for any face F' of $B_{\mathbb{X}}$, $F \subset F'$ implies $F = F'$. For any face F of $B_{\mathbb{X}}$ and any $x^* \in S_{\mathbb{X}^*}$, we say that x^* supports F if $x^*(x) = 1 \ \forall x \in F$. We use the notation $int(F)$ to denote the relative interior of a face F endowed with the subspace topology of F.

For any element $x \in \mathbb{X}$, and any subspace $\mathbb{Y}$ of $\mathbb{X}$, $y_0 \in \mathbb{Y}$ is said to be a best coapproximation (see [15]) to x out of $\mathbb{Y}$ if $\|y_0 - y\| \leq \|x - y\|$ for all $y \in \mathbb{Y}$. As mentioned previously, we require the concept of Birkhoff-James orthogonality to gain a better understanding of the best coapproximation problem. Given $x, y \in \mathbb{X}$, we say that x is Birkhoff-James orthogonal [2, 20] to y, written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$, for all $\lambda \in \mathbb{K}$. It is clear that $y_0 \in \mathbb{Y}$ is a best coapproximation to x out of $\mathbb{Y}$ if and only if

$$\mathbb{Y} \perp_B (x - y_0), \ i.e., \ y \perp_B (x - y_0) \ \forall y \in \mathbb{Y}.$$

Given $\epsilon \in [0,1)$ and $x, y \in \mathbb{X}$, x is said to be ϵ-Birkhoff-James orthogonal [10] to y, written as $x \perp_B^{\epsilon} y$, if

$$\|x + \lambda y\| \geq \|x\| - \epsilon\|\lambda y\| \text{ for every } \lambda \in \mathbb{K}.$$

The above definition, in conjunction with the previously mentioned relation between Birkhoff-James orthogonality and the best coapproximation, naturally leads us to the following definition of ϵ-best coapproximation, introduced in [54]:

Let $\epsilon \in [0,1)$. For a subspace $\mathbb{Y}$ and a given $x \in \mathbb{X}$, we say that $y_0 \in \mathbb{Y}$ is an ϵ-best coapproximation to x out of $\mathbb{Y}$ if

$$\mathbb{Y} \perp_B^\epsilon (x - y_0), \ i.e., \ y \perp_B^\epsilon (x - y_0) \, \forall y \in \mathbb{Y}.$$

As noted in [54], the definitions of best coapproximation and ϵ-best coapproximation motivate us to study the following two special types of subspaces of a Banach space:

Definition 5.1. *(i) A subspace $\mathbb{Y}$ of $\mathbb{X}$ is said to be an anti-coproximinal subspace of $\mathbb{X}$ if for any $x \in \mathbb{X} \setminus \mathbb{Y}$, there does not exist a best coapproximation to x out of $\mathbb{Y}$. Equivalently, a subspace $\mathbb{Y}$ is anti-coproximinal in $\mathbb{X}$ if for any nonzero $x \in \mathbb{X}$, $\mathbb{Y} \not\perp_B x$.*
(ii) A subspace $\mathbb{Y}$ of $\mathbb{X}$ is said to be a strongly anti-coproximinal subspace of $\mathbb{X}$ if for any given $x \in \mathbb{X} \setminus \mathbb{Y}$ and for any $\epsilon \in [0, 1)$, there does not exist an ϵ-best coapproximation to x out of $\mathbb{Y}$. Equivalently, a subspace $\mathbb{Y}$ is strongly anti-coproximinal if for any nonzero $x \in \mathbb{X}$ and for any $\epsilon \in [0, 1)$, $\mathbb{Y} \not\perp_B^\epsilon x$.

For any nonempty set K, $\ell_\infty(K, \mathbb{X})$ stands for the space of all bounded functions from K to $\mathbb{X}$. Given a compact Hausdorff topological space K and a Banach space $\mathbb{X}$, we write $C(K, \mathbb{X})$ to denote the space of continuous functions from K to $\mathbb{X}$, endowed with the sup norm, i.e.,

$$C(K, \mathbb{X}) = \{f \mid f : K \to \mathbb{X} \text{ is continuous and } \sup_{k \in K} \|f(k)\| < \infty\}.$$

Given a locally compact Hausdorff space K and a Banach space $\mathbb{X}$, the space $C_0(K, \mathbb{X})$ is the space of continuous functions f having the property that for any $\epsilon > 0$, there exists a compact set $\Gamma \subset K$ such that $\|f(k)\| < \epsilon$, for any $k \in K \setminus \Gamma$. Whenever K is compact, $C_0(K, \mathbb{X}) = C(K, \mathbb{X})$. For a function $f \in C_0(K, \mathbb{X})$, the norm attainment set of f, denoted by M_f, is defined as $M_f = \{k \in K : \|f(k)\| = \|f\|\}$. Whenever $\mathbb{X} = \mathbb{R}$ or $\mathbb{C}$, we use the standard notations $C(K)$ and $C_0(K)$ in place of $C(K, \mathbb{X})$ and $C_0(K, \mathbb{X})$, respectively. Note that for any $f \in S_{C_0(K, \mathbb{X})}$, M_f is non-empty and compact.

The present chapter is divided into three sections, including the introductory one. In the preliminary section, we provide a characterization of approximate Birkhoff-James orthogonality in the function space as well as in the sequence spaces. First we study the spaces $\ell_\infty(K)$ and $C_0(K)$, and find computationally effective characterizations of these special types of subspaces. We show that the notions of anti-coproximinality and strong anti-coproximinality coincide in both $\ell_\infty(K)$ and $C_0(K)$. Moreover, we study these subspaces in sequence spaces, c_0, c and ℓ_∞. In the last part, we study the stability of these two notions in the space of all vector valued continuous functions.

5.2 Preliminaries

Throughout this chapter, we require several results involving Birkhoff-James orthogonality and its approximate version. Therefore, mainly for the convenience of the readers, it is worth mentioning the relevant results in the context of the present study.

Lemma 5.1. *[20, Th. 2.1] Let $\mathbb{X}$ be a Banach space and let $x, y \in \mathbb{X}$. Then $x \perp_B y$ if and only if there exists $x^* \in J(x)$ such that $x^*(y) = 0$.*

Lemma 5.2. *[12, Th. 2.3] Let $\mathbb{X}$ be a Banach space. Suppose $\epsilon \in [0, 1)$ and let $x, y \in \mathbb{X}$. Then $x \perp_B^\epsilon y$ if and only if there exists $x^* \in J(x)$ such that $|x^*(y)| \leq \epsilon\|y\|$.*

A more general version of Lemma 5.1 has been obtained in [29, Cor. 2.5] as follows:

Theorem 5.1. *Let $\mathbb{X}$ be a Banach space and let $x \in S_{\mathbb{X}}$. Suppose $C \subseteq \mathbb{X}^*$ is such that $B_{\mathbb{X}^*} = \overline{co(C)}^{w^*}$. Then for any $y \in \mathbb{X}$, $x \perp_B y$ if and only if*

$$0 \in co(\{\lim x_n^*(y) : x_n^* \in C \text{ and } \lim x_n^*(x) = 1\}).$$

Following the same technique, a generalized version of Lemma 5.2 can be derived as follows:

Theorem 5.2. *Let $\mathbb{X}$ be a Banach space and let $x, y \in S_{\mathbb{X}}$. Suppose $C \subset B_{\mathbb{X}^*}$ is such that $B_{\mathbb{X}^*} = \overline{co(C)}^{w^*}$. Then the following are equivalent:*

(i) $x \perp_B^\epsilon y$

(ii) $co(\{\lim x_n^*(y) : x_n^* \in C\ \forall n \in \mathbb{N}, \lim x_n^*(x) = 1\}) \cap \mathcal{D}(\epsilon) \neq \emptyset$.

Proof. From Lemma 5.2, $x \perp_B^\epsilon y$ if and only if there exists $x^* \in J(x)$ such that $x^*(y) \in \mathcal{D}(\epsilon)$. Clearly, this is equivalent to $V(\mathbb{X}, x, y) \cap \mathcal{D}(\epsilon) \neq \emptyset$, where $V(\mathbb{X}, x, y) = \{\phi(y) : \phi \in S_{\mathbb{X}^*}, \phi(x) = 1\}$. Now from [29, Th. 2.3] we note that

$$V(\mathbb{X}, x, y) = co(\{\lim x_n^*(y) : x_n^* \in C\ \forall n \in \mathbb{N}, \lim x_n^*(x) = 1\}),$$

thereby finishing the proof. □

Applying the above result, it is rather easy to characterize the approximate Birkhoff-James orthogonality in $\ell_\infty(K, \mathbb{X})$ and in $C_0(K, \mathbb{X})$. Let us first note the following characterization of Birkhoff-James orthogonality.

Theorem 5.3. *[29, Th. 3.2] Let K be a non-empty set and let $\mathbb{X}$ be a Banach space. Let $C \subset S_{\mathbb{X}^*}$ be such that $B_{\mathbb{X}^*} = \overline{co(C)}^{w^*}$. Suppose that $f, g \in \ell_\infty(K, \mathbb{X})$. Then $f \perp_B g$ if and only if*

$$0 \in co\Bigg(\Bigg\{ \lim y_n^*(g(k_n)) : k_n \in K, y_n^* \in C, \ \forall n \in \mathbb{N}, \lim y_n^*(f(k_n)) = \|f\| \Bigg\}\Bigg).$$

Theorem 5.4. *[29, Th. 3.5] Let K be a locally compact Hausdorff space and let $\mathbb{X}$ be a Banach space. Suppose $C \subset S_{\mathbb{X}^*}$ is such that $B_{\mathbb{X}^*} = \overline{co(C)}^{w^*}$. Then for $f, g \in C_0(K, \mathbb{X})$,*

$$f \perp_B g \iff 0 \in co\Bigg(\Bigg\{ y^*(g(k)) : k \in K, y^* \in C, y^*(f(k)) = \|f\| \Bigg\}\Bigg).$$

Although in [29, Th. 3.5], the characterization has been given in $C(K, \mathbb{X})$ space considering K to be compact Hausdorff, we can replicate the arguments for the space $C_0(K, \mathbb{X})$, where K is a locally compact Hausdorff space. In this context, the crucial thing to observe is that for any $f \in C_0(K, \mathbb{X})$, M_f is non-empty and compact.

Using similar technique as in Theorem 3.2 and Theorem 3.5 of [29], and using Theorem 5.2, we obtain the following characterizations of approximate Birkhoff-James orthogonality:

Theorem 5.5. *Let K be a non-empty set and let $\mathbb{X}$ be a Banach space. Suppose that $C \subset S_{\mathbb{X}^*}$ is such that $B_{\mathbb{X}^*} = \overline{co(C)}^{w^*}$. Then for $f, g \in \ell_\infty(K, \mathbb{X})$, $f \perp_B^\epsilon g$ if and only if*

$$co\Bigg(\Bigg\{ \lim y_n^*(g(k_n)) : k_n \in K, y_n^* \in C, \ \forall n \in \mathbb{N}, \lim y_n^*(f(k_n)) = \|f\| \Bigg\}\Bigg) \cap \mathcal{D}(\epsilon\|g\|) \neq \emptyset.$$

Theorem 5.6. *Let K be a locally compact Hausdorff space and let $\mathbb{X}$ be a Banach space. Suppose $C \subset S_{\mathbb{X}^*}$ is such that $B_{\mathbb{X}^*} = \overline{co(C)}^{w^*}$. Then for $f, g \in C_0(K, \mathbb{X})$,*

$$f \perp_B^\epsilon g \iff co\Bigg(\Bigg\{ y^*(g(k)) : k \in K, y^* \in C, y^*(f(k)) = \|f\| \Bigg\}\Bigg) \cap \mathcal{D}(\epsilon\|g\|) \neq \emptyset.$$

Applying Theorem 5.5, we characterize the approximate Birkhoff-James orthogonality in the space $\ell_\infty(K)$.

Theorem 5.7. *Let K be a non-empty set and let $f, g \in S_{\ell_\infty(K)}$. Then $f \perp_B^\epsilon g$ if and only if*

$$co\Bigg(\Bigg\{ \lim \overline{\gamma_n} g(k_n) : k_n \in K, \gamma_n \in \mathbb{K}, |\gamma_n| = 1 \ \forall n \in \mathbb{N}, \lim \gamma_n f(k_n) = 1 \Bigg\}\Bigg) \cap \mathcal{D}(\epsilon) \neq \emptyset.$$

In particular, $f \perp_B g$ if and only if

$$0 \in co\Bigg(\Bigg\{ \lim \overline{\gamma_n} g(k_n) : k_n \in K, \gamma_n \in \mathbb{K}, |\gamma_n| = 1 \ \forall n \in \mathbb{N}, \lim \gamma_n f(k_n) = 1 \Bigg\}\Bigg).$$

We end this section with the following classical result from point-set topology which will be used repeatedly in the later parts.

Lemma 5.3. *(Uryshon's lemma) [30] Let K be a normal space. Let A and B be disjoint closed subsets of K. Then there exists a continuous map $f : K \to [0,1]$ such that $f(x) = 0$ for every $x \in A$, and $f(x) = 1$ for every $x \in B$.*

5.3 Anti-coproximinality in the space of all continuous functions

5.3.1 Anti-coproximinality and strongly anti-coproximinality in $C(K)$

In this section, we consider anti-coproximinal subspaces of the space of all scalar valued bounded (continuous) functions. Unlike the geometric conditions obtained for anti-coproximinal and strongly anti-coproximinal subspaces in the previous chapter, we conduct an analytic study of such subspaces in the space $\ell_\infty(K)$ and $C(K)$. We first characterize the anti-coproximinal and strongly anti-coproximinal subspaces in $\ell_\infty(K)$.

Theorem 5.8. *Let K be a nonempty set and let $\mathbb{Y}$ be a proper closed subspace of $\ell_\infty(K)$. Then the following are equivalent:*

(i) $\mathbb{Y}$ is strongly anti-coproximinal in $\ell_\infty(K)$.

(ii) $\mathbb{Y}$ is anti-coproximinal in $\ell_\infty(K)$.

(iii) for any $k \in K$, there exists $f \in \mathbb{Y}$ such that $|f(k)| > \lim_{n\to\infty} |f(k_n)|$, $\forall \{k_n\} \subset K$ with $k_n \neq k$ for all but finitely many $n \in \mathbb{N}$.

Proof. We begin the proof by noting that (i) $\implies$ (ii) holds trivially. Next we prove (ii) $\implies$ (iii). Suppose on the contrary that there exists $k_0 \in K$ such that for any $f \in \mathbb{Y}, \lim |f(k_n)| \geq |f(k_0)|$, for some $\{k_n\} \subset K$ satisfying $k_n \neq k_0$, for all but finitely many $n \in \mathbb{N}$. Define $g : K \to \mathbb{K}$ such that

$$g(k_0) = 1 \text{ and } g(k) = 0, \ \forall k \in K \setminus \{k_0\}.$$

Clearly $\|g\| = 1$. Let $f \in \mathbb{Y}$ and consider $\{k_n\} \subset K$ such that $k_n \neq k_0$, for all but finitely many $n \in \mathbb{N}$ and $\lim |f(k_n)| = \|f\|$. Since $g(k_n) = 0$, for all but finitely many $n \in \mathbb{N}$, using Theorem

5.7, we obtain $f \perp_B g$. Therefore, $\mathbb{Y} \perp_B g$. In other words, 0 is the best coapproximation to g out of $\mathbb{Y}$, which is clearly a contradiction.

Let us now prove (iii) $\implies$ (i), to finish the proof of the theorem. Suppose on the contrary that $\mathbb{Y}$ is not strongly anti-coproximinal in $\ell_\infty(K)$. Then there exists a non-zero $h \in S_{\ell_\infty(K)}$ and $\epsilon \in [0,1)$ such that $\mathbb{Y} \perp_B^\epsilon h$. For each $k \in K$, take $\widetilde{f}_k \in S_{\mathbb{Y}}$ such that $1 = |\widetilde{f}_k(k)| > \lim |\widetilde{f}_k(k_n)|$, $\forall \{k_n\} \subset K$ satisfying $k_n \neq k$, for all but finitely many $n \in \mathbb{N}$. As $\widetilde{f}_k \perp_B^\epsilon h$, using Theorem 5.7,

$$co\Bigg(\bigg\{\lim \gamma_n h(k_n) : k_n \in K, \gamma_n \in \mathbb{K}, |\gamma_n| = 1\ \forall n \in \mathbb{N}, \lim \gamma_n \widetilde{f}_k(k_n) = 1\bigg\}\Bigg) \cap \mathcal{D}(\epsilon) \neq \emptyset. \tag{5.1}$$

As $1 = |\widetilde{f}_k(k)| > \lim |\widetilde{f}_k(k_n)|$, $\forall \{k_n\} \subset K$ satisfying $k_n \neq k$, for all but finitely many $n \in \mathbb{N}$, it is immediate that (5.1) is equivalent to

$$co\Bigg(\bigg\{\lim \gamma_n h(k) : \gamma_n \in \mathbb{K}, |\gamma_n| = 1\ \forall n \in \mathbb{N}, \lim \gamma_n \widetilde{f}_k(k) = 1\bigg\}\Bigg) \cap \mathcal{D}(\epsilon) \neq \emptyset. \tag{5.2}$$

It is straightforward to see that

$$\{\lim \gamma_n h(k) : \gamma_n \in \mathbb{K}, |\gamma_n| = 1\ \forall n \in \mathbb{N}, \lim \gamma_n \widetilde{f}_k(k) = 1\} = \{\overline{\widetilde{f}_k(k)} h(k)\}.$$

This implies that $\overline{\widetilde{f}_k(k)} h(k) \in \mathcal{D}(\epsilon) \implies |h(k)| \leq \epsilon$. Observe that

$$1 = \|h\| = \sup_{k \in K} |h(k)| \leq \epsilon < 1,$$

a contradiction.

□

We next show that $C_0(K)$ is strongly anti-coproximinal in $\ell_\infty(K)$, whenever K is perfectly normal. Recall that a topological space K is perfectly normal if K is normal and every closed set of K is a G_δ set.

Corollary 5.1. *Let K be locally compact perfectly normal space. Then $C_0(K)$ is strongly anti-coproximinal in $\ell_\infty(K)$.*

Proof. Let $k \in K$ and let U be an open set containing k. Using the Uryshon's Lemma, there exists a continuous function $f : K \to [0,1]$ such that $f^{-1}(1) = \{k\}$ and $f^{-1}(0) = K \setminus U$. Clearly, $f \in C_0(K)$ and f satisfies the condition (iii) of Theorem 5.8. This finishes the proof. □

In [33, Cor. 3.2], Papini et al. gave a sufficient condition for anti-coproximinal subspaces in $C(K)$. In the following theorem we provide a necessary condition for the same.

Proposition 5.1. *Let $\mathbb{Y}$ be an anti-coproximinal subspace of $C_0(K)$, where K is a locally compact normal space. Then for each non-empty open subset U of K, there exists an $f \in \mathbb{Y}$ such that $M_f \subset U$.*

Proof. Suppose on the contrary that there exists a non-empty open set $U \subset K$ such that for any $f \in \mathbb{Y}$, $M_f \cap (K \setminus U) \neq \emptyset$. In other words, for any $f \in \mathbb{Y}$ there exists $k_f \in K \setminus U$ such that $|f(k_f)| \geq |f(k)|,\ \forall k \in K \setminus \{k_f\}$. Let $k_0 \in U$. Using Lemma 5.3 there exists a continuous function $g : K \to [0,1]$ such that $g(k_0) = 1$ and $g(k) = 0,\ \forall k \in K \setminus U$. Clearly, $g \in C_0(K)$. Let $f \in \mathbb{Y}$. Since $\|f\| = |f(k_f)|$, we infer that $\delta_{k_f} \in J(f)$, where $\delta_{k_f} : C_0(K) \to \mathbb{K}$ is given by $\delta_{k_f}(h) = h(k_f),\ \forall h \in C_0(K)$. As $k_f \in K \setminus U$, we note that $\delta_{k_f}(g) = g(k_f) = 0$. So, using Lemma 5.1, we get $f \perp_B g$. Hence $\mathbb{Y} \perp_B g$. This contradicts the fact that $\mathbb{Y}$ is anti-coproximinal in $C_0(K)$. □

An immediate corollary of the above theorem is given below.

Corollary 5.2. *Let K be a locally compact normal space and let $\mathbb{Y}$ be a proper closed subspace of $C_0(K)$. Suppose that there exists an element $k_0 \in K$ such that for any $f \in \mathbb{Y}$, $f(k_0) = 0$. Then $\mathbb{Y}$ is not anti-coproximinal in $C_0(K)$.*

We next provide a sufficient condition for strongly anti-coproximinal subspaces.

Proposition 5.2. *Let $\mathbb{Y}$ be a proper closed subspace of $C_0(K)$, where K is a locally compact Hausdorff space. Suppose that $D \subset K$ is dense in K. If for each $k \in D$, there exists an $f \in \mathbb{Y}$ such that $|f(k)| > |f(k')|$, for all $k' \in K \setminus \{k\}$, then $\mathbb{Y}$ is strongly anti-coproximinal in $C_0(K)$.*

Proof. Suppose on the contrary that $\mathbb{Y}$ is not strongly anti-coproximinal in $C_0(K)$. Then there exists $\epsilon \in [0,1), g \in C_0(K) \setminus \mathbb{Y}$ such that $\mathbb{Y} \perp_B^{\epsilon} g$. Let $k \in D$ and let $\widetilde{f_k} \in \mathbb{Y}$ such that $|\widetilde{f_k}(k)| > |\widetilde{f_k}(k')|,\ \forall k' \in K \setminus \{k\}$. So, $\|\widetilde{f_k}\| = |\widetilde{f_k}(k)|$. It is easy to observe that $J(\widetilde{f_k}) = \{\delta_k\}$, where $\delta_k : C_0(K) \to \mathbb{K}$ is given by $\delta_k(h) = h(k),\ \forall h \in C_0(K)$. As $\widetilde{f_k} \perp_B^{\epsilon} g$, using Lemma 5.2, we get

$$|\delta_k(g)| \leq \epsilon\|g\| \implies |g(k)| \leq \epsilon\|g\|.$$

Observe that

$$\|g\| = \sup_{k \in K} |g(k)| = \sup_{k \in D} |g(k)| \leq \epsilon\|g\| < \|g\|,$$

a contradiction. Thus, $\mathbb{Y}$ is not strongly anti-coproximinal. □

In addition, if we assume that K is locally connected, then we can prove that the above necessary condition given in Proposition 5.1 is also sufficient for anti-coproximinal as well as for strongly anti-coproximinal subspaces. From now on we consider the notation $C_0(K)$ for the space $C_0(K, \mathbb{R})$. We require the following classical result for our purpose.

Lemma 5.4. *[12, Th. 2.4] Let $\mathbb{X}$ be a real Banach space. Suppose $x, y \in \mathbb{X}$ and $\epsilon \in [0,1)$. Then $x \perp_B^\epsilon y$ if and only if there exists $\phi, \psi \in Ext(B_{\mathbb{X}^*}) \cap J(x)$ such that $|((1-t)\phi + t\psi)y| \leq \epsilon\|y\|$, for some $0 \leq t \leq 1$.*

Theorem 5.9. *Let K be a locally connected and locally compact normal space. Suppose that $\mathbb{Y}$ is a proper closed subspace of $C_0(K)$. Then the following are equivalent:*

(i) $\mathbb{Y}$ is strongly anti-coproximinal in $C_0(K)$.

(ii) $\mathbb{Y}$ is anti-coproximinal in $C_0(K)$.

(iii) For each nonempty open subset U of K, there exists an $f \in \mathbb{Y}$ such that $M_f \subset U$.

Proof. We begin this proof by noting that (i) $\implies$ (ii) holds trivially. Also, (ii) $\implies$ (iii) follows from Proposition 5.1.
Therefore, we only prove that (iii) $\implies$ (i). Suppose on the contrary that there exists $\epsilon \in [0,1), g \in C_0(K) \setminus \mathbb{Y}$ such that $\mathbb{Y} \perp_B^\epsilon g$. As M_g is nonempty, suppose $|g(k')| = \|g\|$, for some $k' \in K$. As g is continuous, there exists an open set W containing k' such that $|g(w)| > \epsilon\|g\|$, for all $w \in W$. Since K is locally connected, it follows that there exists an open connected set V such that $k' \in V \subset W$. Then clearly, $|g(v)| > \epsilon\|g\|$, for all $v \in V$. Now, from (iii), there exists an $f \in \mathbb{Y}$ such that $M_f \subset V$. Since $f \perp_B^\epsilon g$, using Lemma 5.4, there exist $\phi, \psi \in Ext(B_{C_0(K)^*}) \cap J(f)$ such that

$$|((1-t)\phi + t\psi)(g)| \leq \epsilon\|g\|,$$

for some $t \in [0,1]$. It is well known that (cf. [44, Chapter I, 1.10]) $\phi = \delta_{k_1}$ and $\psi = \delta_{k_2}$ for some $k_1, k_2 \in M_f$, where $\delta_{k_i} : C_0(K) \to \mathbb{R}$ such that $\delta_{k_i}(h) = h(k_i),\ \forall h \in C_0(K)$. Therefore, $|((1-t)\delta_{k_1} + t\delta_{k_2})(g)| \leq \epsilon\|g\|$. So, $|(1-t)g(k_1) + tg(k_2)| \leq \epsilon\|g\|$. As $k_1, k_2 \in V$, V is connected and g is continuous, it is straightforward to see that there exists $k_0 \in V$ such that $|g(k_0)| \leq \epsilon\|g\|$, a contradiction. So $\mathbb{Y}$ is strongly anti-coproximinal in $C_0(K)$, as desired. □

Remark 5.10. *It is well known that $\mathbb{X}^*$ is embedded into $C(S_{\mathbb{X}})$ via the map $\Psi : z^* \longrightarrow z^*|_{S_{\mathbb{X}}}$. Whenever $\mathbb{X}$ is finite-dimensional, we consider the subspace $\Psi(\mathbb{X}^*)$ in $C(S_{\mathbb{X}})$. Note that for any $z^* \in \mathbb{X}^*$, M_{z^*} always contains the antipodal points of $S_{\mathbb{X}}$. Therefore, applying the condition (iii) of Theorem 5.9, we can observe that $\Psi(\mathbb{X}^*)$ is not anti-coproximinal in $C(S_{\mathbb{X}})$.*

Next we show that any finite-codimensional subspace of $C_0(K)$ is strongly anti-coproximinal, whenever K has additional nice properties.

Theorem 5.11. *Let K be a locally connected, locally compact and perfectly normal space such that K does not contain any isolated point. Then any finite-codimensional subspace of $C_0(K)$ is strongly anti-coproximinal.*

Proof. Let $\mathbb{Y}$ be an m-codimensional subspace of $C_0(K)$. Suppose on the contrary that $\mathbb{Y}$ is not strongly anti-coproximinal in $C_0(K)$. Applying Theorem 5.9, there exists an nonempty open set $U \subset K$ such that for any $f \in \mathbb{Y}, M_f \cap (K \setminus U) \neq \emptyset$. So, for any $f \in \mathbb{Y}$, take $k_f \in K \setminus U$ such that $\|f\| = |f(k_f)|$. Let $k_1, k_2, \ldots, k_{m+1} \in U$. As K is Hausdorff, let $V_1, V_2, \ldots, V_m, V_{m+1} \subset U$ be the open sets of K such that $k_i \in V_i,\ V_i \cap V_j = \emptyset, \forall i, j \in \{1, 2, \ldots, m+1\}, i \neq j$. As K is perfectly normal, using Lemma 5.3, we assert that there exist continuous functions $g_i : K \to [0, 1]$ such that

$$g_i^{-1}(\{1\}) = \{k_i\} \text{ and } g_i^{-1}(\{0\}) = K \setminus V_i, \forall\, i = 1, 2, \ldots, m+1.$$

Clearly, for each $1 \leq i \leq m+1$, $M_{g_i} = \{k_i\}$ and so, $g_i \in C_0(K) \setminus \mathbb{Y}$. Moreover it is obvious that $g_i(k_j) = 0$, whenever $i \neq j$. Observe that $\{g_1, g_2, \ldots, g_{m+1}\}$ is a linearly independent set. Since $\mathbb{Y}$ is an m-codimensional subspace of $C_0(K)$, so

$$g_{m+1} = \alpha_1 g_1 + \alpha_2 g_2 + \ldots + \alpha_m g_m + f, \text{where} \quad \alpha_i \in \mathbb{K}, f \in \mathbb{Y}.$$

Since $k_f \in K \setminus U$ such that $\|f\| = |f(k_f)|$, we have

$$g_{m+1}(k_f) = \sum_{i=1}^{m} \alpha_i g_i(k_f) + f(k_f) \implies f(k_f) = 0.$$

As $\|f\| = |f(k_f)|$, this implies $f = 0$, therefore, $g_{m+1} = \sum_{i=1}^{m} \alpha_i g_i$, a contradiction. This establishes the theorem. □

Remark 5.12. *While studying the specialty of inner product spaces among Banach spaces, James stated an interesting result in [19, p. 564]:* "For no hyperspace H of the space $C[a,b]$ of continuous functions defined on $[a, b] \subset \mathbb{R}$ there is an element $f \in C[a, b]$ with $H \perp_B f$. " *Observe that this is the least favorable case for the space to be an inner product space from the perspective of [19, Th. 4], which states that a Banach space $\mathbb{X}$ is an inner product space if and only if for a given hyperspace H there exists an element $z \in \mathbb{X}$ such that $H \perp_B z$. To the best of our knowledge, $C[a, b]$ is the only known example of a Banach space with such a property. Following Theorem 5.11, we can strengthen the statement above given by James in the following way:*

Whenever K is locally connected, compact and perfectly normal space, there is no subspace $\mathbb{Y}$ of $C(K)$ with finite-codimension such that given any $\epsilon \in [0, 1)$, there is an element $f \in C(K)$ satisfying $\mathbb{Y} \perp_B^{\epsilon} f$.

5.3.2 Anti-coproximinal and strongly anti-coproximinality in the sequence spaces

The following result is an easy consequence of Theorem 5.8.

Corollary 5.3. *Let $\mathbb{Y}$ be a subspace of ℓ_∞. Then the following are equivalent:*

(i) $\mathbb{Y}$ *is strongly anti-coproximinal in* ℓ_∞.

(ii) $\mathbb{Y}$ *is anti-coproximinal in* ℓ_∞.

(iii) for each $r \in \mathbb{N}$, *there exists* $\widetilde{y} = (y_1, y_2, \ldots) \in \mathbb{Y}$ *such that*

(*a*) $|y_r| > |y_n|\ \forall n \in \mathbb{N} \setminus \{r\}$,

(*b*) $|y_r| > \lim |y_{n_k}|$, *for any sequence* $\{n_k\}_{k\in\mathbb{N}}$ *satisfying* $n_k \neq r$ *for all but finitely many* $k \in \mathbb{N}$.

It is clear from Corollary 5.3 that c_0 is a strongly anti-coproximinal subspace of ℓ_∞.

We next characterize anti-coproximinal and strongly anti-coproximinal subspaces in c_0 and c.

Proposition 5.3. *Let $\mathbb{X} = c_0$ or c and let $\mathbb{Y}$ be a subspace of $\mathbb{X}$. Then the following are equivalent:*

(i) $\mathbb{Y}$ *is strongly anti-coproximinal in* $\mathbb{X}$.

(ii) $\mathbb{Y}$ *is anti-coproximinal in* $\mathbb{X}$.

(iii) for any $r \in \mathbb{N}$, *there exists* $\widetilde{y} = (y_1, y_2, \ldots) \in \mathbb{Y}$ *such that* $|y_r| > |y_n|,\ \forall n \in \mathbb{N} \setminus \{r\}$.

Proof. Clearly, (i) $\implies$ (ii) holds trivially. We prove that (ii) $\implies$ (iii). Suppose on the contrary that there exists $r_0 \in \mathbb{N}$ such that for any $\widetilde{y} = (y_1, y_2, \ldots) \in \mathbb{Y}$, $|y_{r_0}| \leq |y_s|$, for some $s \neq r_0$. Take $x = (x_1, x_2, \ldots)$ such that $x_{r_0} = 1$ and $x_n = 0\ \forall n \in \mathbb{N} \setminus \{r_0\}$. Let $y = (y_1, y_2, \ldots) \in \mathbb{Y}$. If $\lim |y_n| \neq \|y\|$, then there exists $k \in \mathbb{N}$ such that $k \neq r_0$ and $|y_k| = \|y\|$. Since $x_k = 0$ for all $k \neq r_0$, we get $0 \in co(\{\overline{y_n} x_n : |y_n| = \|y\|\})$, therefore following [7, Th. 2.12], $y \perp_B x$. Now consider the case that $\lim |y_n| = \|y\|$. As $\lim |x_n| = 0$, we have $0 \in co(\{\overline{y_n} x_n : |y_n| = \|y\|\} \cup \{\lim \overline{y_n} x_n\})$. Again following [7, Th. 2.12], $y \perp_B x$. This implies that $\mathbb{Y} \perp_B x$. This contradicts that $\mathbb{Y}$ is anti-coproximinal in $\mathbb{X}$.

We next prove that (iii) $\implies$ (i). Suppose on the contrary that there exists an $\epsilon \in [0, 1)$ and a nonzero $x = (x_1, x_2, \ldots) \in \mathbb{X}$ such that $\mathbb{Y} \perp_B^\epsilon x$. Let $n \in \mathbb{N}$. There exists $\widetilde{y_n} = (y_1, y_2, \ldots) \in \mathbb{Y}$

such that $|y_n| > |y_i|, \ \forall i \in \mathbb{N} \setminus \{n\}$. Let $e_n^* \in \mathbb{X}^*$ be such that $e_n^*(u_1, u_2, \ldots) = u_n, \ \forall (u_1, u_2, \ldots) \in \mathbb{X}$. It is clear that $J(\widetilde{y_n}) = \{e_n^*\}$. Since $\widetilde{y_n} \perp_B^{\epsilon} x$, applying Lemma 5.2, we get $|e_n^*(x)| \leq \epsilon \|x\|$. This implies $|x_n| \leq \epsilon \|x\|$. Observe that

$$\|x\| = \sup_{n \in \mathbb{N}} |x_n| \leq \epsilon \|x\| < \|x\|.$$

This is a contradiction. Hence the theorem. □

Our next result shows that $\ell_\infty(\mathbb{R})$ does not admit a finite-dimensional polyhedral subspace which is anti-coproximinal.

Proposition 5.4. *Let $\mathbb{X} = \ell_\infty(\mathbb{R})$. Any finite-dimensional polyhedral subspace of $\mathbb{X}$ is not anti-coprxominal.*

Proof. Let $\mathbb{Y}$ be a finite-dimensional polyhedral subspace of $\ell_\infty(\mathbb{R})$. Suppose on the contrary that $\mathbb{Y}$ is anti-coproximinal in $\mathbb{X}$. From Corollary 5.3, we see that for any $n \in \mathbb{N}$, there exists $\widetilde{y_n} = (y_1, y_2, \ldots) \in \mathbb{Y}$ such that $|y_n| > |y_i| \ \forall i \in \mathbb{N} \setminus \{n\}$. For each $n \in \mathbb{N}$, define

$$F_n = \{x = (x_1, x_2, \ldots) \in \mathbb{X} : x_n = 1, |x_i| \leq 1 \ \forall i \in \mathbb{N} \setminus \{n\}\}.$$

Clearly, $F_n \subset S_{\mathbb{X}}$. We claim that F_n is a face of $B_{\mathbb{X}}$. Let $\widetilde{u} = (u_1, u_2, \ldots), \widetilde{w} = (w_1, w_2, \ldots) \in S_{\mathbb{X}}$ be such that $(1-t)\widetilde{u} + t\widetilde{w} \in F_n$. Clearly, $|u_i|, |w_i| \leq 1$. So,

$$(1-t)u_n + tw_n = 1 \implies u_n = w_n = 1.$$

Therefore, $\widetilde{u}, \widetilde{w} \in F_n$. This implies F_n is a face of $B_{\mathbb{X}}$. Let $F_n' = F_n \cap B_{\mathbb{Y}}$. Clearly, $\widetilde{y_n} \in F_n'$ and F_n' is a face of $B_{\mathbb{Y}}$. This implies that $B_{\mathbb{Y}}$ contains infinitely many faces, which contradicts the fact that $\mathbb{Y}$ is a polyhedral Banach space. This completes the proof.

□

Remark 5.13. *Similarly, using Proposition 5.3, we can show that the space c_0 and c do not contain a finite-dimensional polyhedral subspace which is anti-coproximinal.*

It is well known that every finite-dimensional subspace of c_0 is polyhedral. Thus the following corollary is immediate.

Corollary 5.4. *Any finite-dimensional subspace of c_0 is not anti-coprxominal.*

Unlike c_0, we give an example of a finite-dimensional strongly anti-coproximinal subspace of c and ℓ_∞.

Example 5.14. *Let* $\mathbb{X} = c$ *or* ℓ_∞. *Suppose*

$$\begin{aligned}\widetilde{u} &= (\cos\frac{\pi}{2}, \cos\frac{\pi}{4}, \ldots, \cos\frac{\pi}{2n}, \ldots),\\ \widetilde{v} &= (\sin\frac{\pi}{2}, \sin\frac{\pi}{4}, \ldots, \sin\frac{\pi}{2n}, \ldots).\end{aligned}$$

Let $\mathbb{Y} = span\{\widetilde{u}, \widetilde{v}\}$. *For any* $n \in \mathbb{N}$, *take* $\widetilde{y} = \cos\frac{\pi}{2n}\widetilde{u} + \sin\frac{\pi}{2n}\widetilde{v}$. *It is easy to observe that* $|y_n| > |y_i|\ \forall i \in \mathbb{N} \setminus \{n\}$, *where* $\widetilde{y} = (y_1, y_2, \ldots)$. *So, using Proposition 5.3 and Corollary 5.3, we conclude that* $\mathbb{Y}$ *is strongly anti-coproximinal in* $\mathbb{X}$, *whereas from Proposition 5.4, it follows that* $\mathbb{Y}$ *is not polyhedral.*

Though c_0 does not have any finite-dimensional strongly anti-coproximinal subspace (see Corollary 5.4), in the following example we consider an infinite-dimensional subspace which is strongly anti-coproximinal in c_0.

Example 5.15. *Let* $\mathbb{Y} = \{(x_1, x_2, \ldots) \in c_0 : x_1 + x_2 + x_3 = 0\}$. *Let*

$$\begin{aligned}\widetilde{y_1} &= (1, -\frac{1}{2}, -\frac{1}{2}, 0, \ldots, 0)\\ \widetilde{y_2} &= (-\frac{1}{2}, 1, -\frac{1}{2}, 0, \ldots, 0)\\ \widetilde{y_3} &= (-\frac{1}{2}, -\frac{1}{2}, 1, 0, \ldots, 0)\\ \widetilde{y_i} &= (0, 0, \ldots, 0, \underset{i-th}{1}, 0, \ldots), \quad \forall i > 3,\end{aligned}$$

Clearly, $\widetilde{y_n} \in \mathbb{Y}$, *for each* $n \in \mathbb{N}$. *It is immediate that for any* $n \in \mathbb{N}$, $\widetilde{y_n}$ *satisfies the sufficient condition of Proposition 5.3. So,* $\mathbb{Y}$ *is strongly anti-coproximinal in* c_0.

5.4 Anti-coproximinality in the space $C(K, \mathbb{X})$

Our final goal in this section is to investigate whether $C(K, \mathbb{Y})$ is anti-coproximinal (strongly anti-coproximinal) in $C(K, \mathbb{X})$ if $\mathbb{Y}$ is anti-coproximinal (strongly anti-coproximinal) in $\mathbb{X}$ and vice versa. We require the following lemma for our purpose.

Lemma 5.5. *Let* K *be a compact perfectly normal space and let* $\mathbb{Y}$ *be a subspace of* $\mathbb{X}$. *Suppose* $f \in S_{C(K,\mathbb{X})}$ *is such that* $C(K, \mathbb{Y}) \perp_B^\epsilon f$, *where* $\epsilon \in [0, 1)$. *Then* $f(M_f) \cap (\mathbb{X} \setminus \mathbb{Y}) \neq \emptyset$.

Proof. Suppose on the contrary that $f(M_f) \subset \mathbb{Y}$. Let $k_0 \in M_f$ and let $U \subset K$ be an open set containing k_0. As K is perfectly normal, there exists a continuous function $\phi : K \to [0, 1]$ such that $\phi^{-1}(\{1\}) = \{k_0\}$ and $\phi^{-1}(\{0\}) = K \setminus U$. Consider the function $g : K \to \mathbb{X}$ such that $g(k) = \phi(k)f(k_0)\ \forall k \in K$. Clearly, g is continuous. As $f(k_0) \in \mathbb{Y}$, it follows trivially that $g(k) \in \mathbb{Y}\ \forall k \in K$. So, $g \in C(K, \mathbb{Y})$. It is easy to see that $M_g = \{k_0\}$. Consider the set

$$S = co\bigg(\{y^*(f(k)) : k \in K, y^* \in C, y^*(g(k)) = \|g\|\}\bigg),$$

where $B_{\mathbb{X}^*} = \overline{co(C)}^{w^*}$. As $M_g = \{k_0\}$, it is easy to observe that $y^*(g(k)) = \|g\|$ implies that $k = k_0$. So,

$$S = co(\{y^*(f(k_0)) : y^* \in C, y^*(g(k_0)) = 1\}).$$

Observe that $f(k_0) = g(k_0)$, so $y^*(f(k_0)) = 1$, where $y^*(g(k_0)) = 1$. Therefore $S = \{1\}$. Now following Theorem 5.6, $g \perp_B^{\epsilon} f$ implies that $S \cap \mathcal{D}(\epsilon) \neq \emptyset$, a contradiction. This completes the proof. □

Theorem 5.16. *Let K be a compact perfectly normal space and let $\mathbb{Y}$ be a subspace of $\mathbb{X}$. Then*

(i) $C(K, \mathbb{Y})$ is strongly anti-coproximinal in $C(K, \mathbb{X})$ if and only if $\mathbb{Y}$ is strongly anti-coproximinal in $\mathbb{X}$.

(ii) $C(K, \mathbb{Y})$ is anti-coproximinal in $C(K, \mathbb{X})$ if and only if $\mathbb{Y}$ is anti-coproximinal in $\mathbb{X}$.

Proof. (i) We first prove the sufficient part. Suppose on the contrary that there exists a nonzero $h \in C(K, \mathbb{X})$ and $\epsilon \in [0, 1)$ such that $C(K, \mathbb{Y}) \perp_B^{\epsilon} h$. Let $\|h\| = 1$. As $h \neq 0$, using Lemma 5.5 there exists $k_0 \in K$ such that $h(k_0) \in S_{\mathbb{X}} \setminus S_{\mathbb{Y}}$. Let $U \subset K$ be an open set containing k_0. As K is perfectly normal, using Lemma 5.3, there exists a continuous function $\phi : K \to [0, 1]$ such that $\phi^{-1}(\{1\}) = \{k_0\}$ and $\phi^{-1}(\{0\}) = K \setminus U$. Let $y \in \mathbb{Y}$. Define $f_y : K \to \mathbb{Y}$ such that $f_y(k) = \phi(k)y\ \forall k \in K$. Clearly, $f_y \in C(K, \mathbb{Y})$ and $f_y(k) = 0,\ \forall k \in K \setminus U$. Moreover, $f_y(k_0) = y$ and for any $k \in K \setminus \{k_0\}$, $\|f_y(k)\| = \|\alpha(k)y\| < \|y\|$. So, $M_{f_y} = \{k_0\}$. As $f_y \perp_B^{\epsilon} h$, using Theorem 5.6,

$$co(\{y^*(h(k_0)) : y^* \in C, y^*(f_y(k_0)) = \|f_y\|\}) \cap \mathcal{D}(\epsilon) \neq \emptyset.$$

Applying Carathéodory's Theorem (see [40, Th. 17.1]), there exists real scalars $t_i (i = 1, 2, 3), t_i > 0, \sum_{i=1}^{3} t_i = 1$ such that

$$\left|\sum_{i=1}^{3} t_i y_i^*(h(k_0))\right| \leq \epsilon,$$

where $y_i^*(f_y(k_0)) = \|f_y\| = \|f_y(k_0)\|$. For each $1 \leq i \leq 3$, $y_i^* \in J(f_y(k_0))$, which implies that $\sum_{i=1}^{3} t_i y_i^* \in J(f_y(k_0))$. Since $\|h(k_0)\| = 1$, it follows that $|\sum_{i=1}^{3} t_i y_i^*(h(k_0))| \leq \epsilon\|h(k_0)\|$ and therefore, following Lemma 5.2, we get that $f_y(k_0) \perp_B^{\epsilon} h(k_0)$. Since $f_y(k_0) = y$, we have $y \perp_B^{\epsilon} h(k_0)$. This implies $\mathbb{Y} \perp_B^{\epsilon} h(k_0)$, which contradicts that $\mathbb{Y}$ is strongly anti-coproximinal in $\mathbb{X}$.

Let us now prove the necessary part. Suppose on the contrary that $\mathbb{Y}$ is not strongly anti-coprximinal in $\mathbb{X}$. Then there exists an $x \in \mathbb{X} \setminus \mathbb{Y}$ and $\epsilon \in [0,1)$ such that $\mathbb{Y} \perp_B^\epsilon x$. Define $g : K \to \mathbb{X}$ such that $g(k) = x \ \forall k \in K$. Let $f \in C(K, \mathbb{Y})$, and let $k \in M_f$. As $f(k) \in \mathbb{Y}$, it is clear that $f(k) \perp_B^\epsilon g(k)$. Observe that

$$\|f + \lambda g\| \geq \|f(k) + \lambda g(k)\| \geq \|f(k)\| - \epsilon|\lambda|\|g(k)\| \geq \|f\| - \epsilon|\lambda|\|g\|,$$

which shows that $f \perp_B^\epsilon g$. However, this implies that $C(K, \mathbb{Y}) \perp_B^\epsilon g$, contradicting that $C(K, \mathbb{Y})$ is strongly anti-coproximinal in $C(K, \mathbb{X})$. This completes the proof of (i).

(ii) Let us first prove the sufficient part. Suppose on the contrary that there exists a nonzero $h \in C(K, \mathbb{X})$ such that $C(K, \mathbb{Y}) \perp_B h$. As $h \neq 0$, then there exists $k_0 \in K$ such that $h(k_0) \in \mathbb{X} \setminus \mathbb{Y}$. For any $y \in \mathbb{Y}$, let $f_y : K \to \mathbb{Y}$ be the same function defined in (i). Using similar arguments as given in the proof of (i), we obtain that $M_{f_y} = \{k_0\}$. As $f_y \perp_B h$, applying [29, Th. 4.3], $f_y(k_0) \perp_B h(k_0)$. Since $f_y(k_0) = y$, we have $y \perp_B h(k_0)$. This implies that $\mathbb{Y} \perp_B h(k_0)$, contradicting that $\mathbb{Y}$ is anti-coproximinal in $\mathbb{X}$.

For the necessary part, suppose on the contrary that $\mathbb{Y}$ is not anti-coproximinal in $\mathbb{X}$. Then there exists an $x \in \mathbb{X} \setminus \mathbb{Y}$ such that $\mathbb{Y} \perp_B x$. Let $g : K \to \mathbb{X}$ be the function defined as $g(k) = x \ \forall k \in K$. Let $f \in C(K, \mathbb{Y})$ and let $k \in M_f$. As $f(k) \in \mathbb{Y}$, $f(k) \perp_B g(k)$. Observe that

$$\|f + \lambda g\| \geq \|f(k) + \lambda g(k)\| \geq \|f(k)\| = \|f\|,$$

proving that $f \perp_B g$. This implies $C(K, \mathbb{Y}) \perp_B g$, a contradiction to the fact that $C(K, \mathbb{Y})$ is anti-coproximinal in $C(K, \mathbb{X})$. □

Remark 5.17. *We note that the notions of anti-coproximinal and strongly anti-coproximinal subspaces coincide in the $C(K, \mathbb{R})$ spaces (see Th. 5.9). On the other hand, these two notions do not coincide in $C(K, \mathbb{X})$, for an arbitrary Banach space $\mathbb{X}$. Indeed, let us take a subspace $\mathbb{Y}$ of $\mathbb{X}$ which is anti-coproximinal but not strongly anti-coproximinal (see [54, Example 2.21]). Then from the above theorem, it follows that $C(K, \mathbb{Y})$ is anti-coproximinal but not strongly anti-coproximinal in $C(K, \mathbb{X})$.*

CHAPTER 6

ANTI-COPROXIMINALITY IN THE SPACE OF ALL BOUNDED LINEAR OPERATORS

6.1 Introduction

Continuing our exploration of anti-coproximinal and strongly anti-coproximinal subspaces, this chapter focuses on their behavior within the space of all bounded linear operators between Banach spaces. We examine the existence and structure of such subspaces in operator spaces, drawing parallels and contrasts with previously studied settings. In particular, we derive necessary and sufficient conditions for the existence of strongly anti-coproximinal subspaces in general Banach space settings. We now mention the basic notations and terminologies that will be used throughout this chapter.

Let $\mathbb{X}, \mathbb{Y}$ denote Banach spaces over the field $\mathbb{K}$, either real or complex. For $\epsilon > 0$, let us set $\mathcal{D}(\epsilon) = \{z \in \mathbb{K} : |z| \leq \epsilon\}$ and let $\mathbb{T}$ denote the unit circle in the complex plane. We

Content of this chapter is based on the following paper:

- Sohel, S., Ghosh, S., Sain, D., Paul, K., *On best coapproximations and some special subspaces of function spaces*, https://doi.org/10.48550/arXiv.2504.13464.

use the notations $B_{\mathbb{X}}$ and $S_{\mathbb{X}}$ to denote the unit ball and unit sphere of $\mathbb{X}$, respectively. Let $\mathbb{L}(\mathbb{X},\mathbb{Y})$ ($\mathbb{K}(\mathbb{X},\mathbb{Y})$) be the space of all bounded (compact) linear operators between $\mathbb{X}$ and $\mathbb{Y}$. The space of all finite-rank operators from $\mathbb{X}$ to $\mathbb{Y}$ is denoted by the notion $\mathcal{F}(\mathbb{X},\mathbb{Y})$. The dual space of $\mathbb{X}$ is denoted by $\mathbb{X}^*$. For a non-zero $x \in \mathbb{X}$, $x^* \in S_{\mathbb{X}^*}$ is said to be a supporting functional at x if $x^*(x) = \|x\|$. The set of all supporting functionals at x is denoted by $J(x)$, i.e., $J(x) = \{x^* \in S_{\mathbb{X}^*} : x^*(x) = \|x\|\}$. A non-zero element $x \in \mathbb{X}$ is said to be smooth if $J(x)$ is a singleton. The collection of all smooth points in $\mathbb{X}$ is denoted by $Sm(\mathbb{X})$. A Banach space $\mathbb{X}$ is said to be smooth if $Sm(\mathbb{X}) = \mathbb{X} \setminus \{0\}$. For a subspace $\mathbb{Y}$ of $\mathbb{X}$, we use the notation $\mathcal{J}_{\mathbb{Y}} = \{y^* \in S_{\mathbb{X}^*} : y^*(y) = 1, \text{for some } y \in Sm(\mathbb{X}) \cap S_{\mathbb{Y}}\}$. It is easy to check that $\mathcal{J}_{\mathbb{Y}} \subseteq Ext(B_{\mathbb{X}^*})$. Whenever $Sm(\mathbb{X}) \cap S_{\mathbb{Y}} = \emptyset$, we define $\mathcal{J}_{\mathbb{Y}} = \emptyset$. The convex hull of a set S is denoted as $co(S)$. For a convex set C, an element $x \in C$ is said to be an extreme point if $x = (1-t)y + tz$, for some $t \in (0,1)$ and some $y, z \in C$ implies that $x = y = z$. The set of all extreme points of C is denoted by $Ext(C)$. A finite-dimensional real Banach space is said to be polyhedral if $B_{\mathbb{X}}$ is a polyhedron, or, equivalently, if $Ext(B_{\mathbb{X}})$ is finite. A convex set $F \subset S_{\mathbb{X}}$ is said to be a face of $B_{\mathbb{X}}$ if for any $y, z \in B_{\mathbb{X}}$, $\frac{1}{2}(y+z) \in F$ implies that $y, z \in F$. F is called a maximal face if for any face F' of $B_{\mathbb{X}}$, $F \subset F'$ implies $F = F'$. For any face F of $B_{\mathbb{X}}$ and any $x^* \in S_{\mathbb{X}^*}$, we say that x^* supports F if $x^*(x) = 1\ \forall x \in F$. We use the notation $int(F)$ to denote the relative interior of a face F endowed with the subspace topology of F. The space $\mathbb{X}$ is said to be strictly convex if $Ext(B_{\mathbb{X}}) = S_{\mathbb{X}}$. An element $x \in S_{\mathbb{X}}$ is said to be rotund if for some $y \in B_{\mathbb{X}}$, $\|\frac{x+y}{2}\| = 2$ implies $x = y$. In a strictly convex space every element of the unit sphere is rotund. An element $x \in S_{\mathbb{X}}$ is said to be weakly almost locally uniformly rotund or w-ALUR point if given any $\{x_m^*\}_{m\in\mathbb{N}} \subset B_{\mathbb{X}^*}$ and any $\{x_n\}_{n\in\mathbb{N}} \subset B_{\mathbb{X}}$,

$$\lim_{m\to\infty} \lim_{n\to\infty} x_m^*\left(\frac{x_n + x}{2}\right) = 1$$

implies that $x_n \xrightarrow{w} x$. The space $\mathbb{X}$ is said to be w-ALUR if each element of $S_{\mathbb{X}}$ is w-ALUR. An element $x \in S_{\mathbb{X}}$ is said to be an exposed point of $B_{\mathbb{X}}$ if there exists $x^* \in J(x)$ such that $x^*(y) < 1 = x^*(x)$, for any $y \in S_{\mathbb{X}} \setminus \{x\}$. Clearly, every exposed point of $B_{\mathbb{X}}$ is also an extreme point of $B_{\mathbb{X}}$. We say $x \in S_{\mathbb{X}}$ to be a strongly exposed point of $B_{\mathbb{X}}$ if there exists $x^* \in J(x)$ such that for any sequence $\{x_n\} \subset B_{\mathbb{X}}$, $x^*(x_n) \longrightarrow 1 = x^*(x)$ implies that $x_n \longrightarrow x$. The set of all exposed points and strongly exposed points of $B_{\mathbb{X}}$ are denoted by $Exp(B_{\mathbb{X}})$ and $st\text{-}Exp(B_{\mathbb{X}})$, respectively.

For any element $x \in \mathbb{X}$, and any subspace $\mathbb{Y}$ of $\mathbb{X}$, $y_0 \in \mathbb{Y}$ is said to be a best coapproximation (see [15]) to x out of $\mathbb{Y}$ if $\|y_0 - y\| \leq \|x - y\|$ for all $y \in \mathbb{Y}$. As mentioned previously, we require the concept of Birkhoff-James orthogonality to gain a better understanding of the best coapproximation problem. Given $x, y \in \mathbb{X}$, we say that x is Birkhoff-James orthogonal [2, 20] to y, written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$, for all $\lambda \in \mathbb{K}$. It is clear that $y_0 \in \mathbb{Y}$ is a best coapproximation to x out of $\mathbb{Y}$ if and only if

$$\mathbb{Y} \perp_B (x - y_0), \ i.e., \ y \perp_B (x - y_0) \ \forall y \in \mathbb{Y}.$$

Given $\epsilon \in [0, 1)$ and $x, y \in \mathbb{X}$, x is said to be ϵ-Birkhoff-James orthogonal [11] to y, written as $x \perp_B^\epsilon y$, if

$$\|x + \lambda y\| \geq \|x\| - \epsilon\|\lambda y\| \text{ for every } \lambda \in \mathbb{K}.$$

The above definition, in conjunction with the previously mentioned relation between Birkhoff-James orthogonality and the best coapproximation, naturally leads us to the following definition of ϵ-best coapproximation, introduced in [54]:

Let $\epsilon \in [0, 1)$. For a subspace $\mathbb{Y}$ and a given $x \in \mathbb{X}$, we say that $y_0 \in \mathbb{Y}$ is an ϵ-best coapproximation to x out of $\mathbb{Y}$ if

$$\mathbb{Y} \perp_B^\epsilon (x - y_0), \ i.e., \ y \perp_B^\epsilon (x - y_0) \ \forall y \in \mathbb{Y}.$$

As noted in [54], the definitions of best coapproximation and ϵ-best coapproximation motivate us to study the following two special types of subspaces of a Banach space:

Definition 6.1. *(i) A subspace $\mathbb{Y}$ of $\mathbb{X}$ is said to be an anti-coproximinal subspace of $\mathbb{X}$ if for any $x \in \mathbb{X} \setminus \mathbb{Y}$, there does not exist a best coapproximation to x out of $\mathbb{Y}$. Equivalently, a subspace $\mathbb{Y}$ is anti-coproximinal in $\mathbb{X}$ if for any nonzero $x \in \mathbb{X}$, $\mathbb{Y} \not\perp_B x$.*
(ii) A subspace $\mathbb{Y}$ of $\mathbb{X}$ is said to be a strongly anti-coproximinal subspace of $\mathbb{X}$ if for any given $x \in \mathbb{X} \setminus \mathbb{Y}$ and for any $\epsilon \in [0, 1)$, there does not exist an ϵ-best coapproximation to x out of $\mathbb{Y}$. Equivalently, a subspace $\mathbb{Y}$ is strongly anti-coproximinal if for any nonzero $x \in \mathbb{X}$ and for any $\epsilon \in [0, 1)$, $\mathbb{Y} \not\perp_B^\epsilon x$.

First we study the anti-coproximinal and the strongly anti-coproximinal subspaces of a Banach space, and obtain some useful necessary and sufficient conditions for the same. In the last part, we consider the space of all bounded operators $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ and investigate when $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ is strongly anti-coproximinal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. We use the so-called *BŠ-Property* of operators to investigate anti-coproximinality of $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. We also prove that the set of norm attaining operators satisfying the *BŠ-Property* is dense in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$, if $\mathbb{X}$ satisfies the Radon-Nikodym Property. Finally, we provide a sufficient condition for strong anti-coproximinality of a subspace in $\mathbb{L}(\mathbb{H})$, for any Hilbert space $\mathbb{H}$.

6.2 Anti-coproximinality in general Banach space

We begin this section with the following lemmas, which we require for the study of anti-coproximinal and strongly anti-coproximinal subspaces.

Lemma 6.1. *[20, Th. 2.1] Let $\mathbb{X}$ be a Banach space and let $x, y \in \mathbb{X}$. Then $x \perp_B y$ if and only if there exists $x^* \in J(x)$ such that $x^*(y) = 0$.*

Lemma 6.2. *[12, Th. 2.3] Let $\mathbb{X}$ be a Banach space. Suppose $\epsilon \in [0, 1)$ and let $x, y \in \mathbb{X}$. Then $x \perp_B^\epsilon y$ if and only if there exists $x^* \in J(x)$ such that $|x^*(y)| \leq \epsilon \|y\|$.*

Lemma 6.3. *[41, Th. 4.7] Let $\mathbb{Z}$ be a subspace of $\mathbb{X}^*$. Then $({}^\perp\mathbb{Z})^\perp = \overline{\mathbb{Z}}^{w^*}$, where ${}^\perp\mathbb{Z} = \{x \in \mathbb{X} : z^*(x) = 0 \ \forall z^* \in \mathbb{Z}\}$ and $({}^\perp\mathbb{Z})^\perp = \{x^* \in \mathbb{X}^* : x^*(z) = 0 \ \forall z \in {}^\perp\mathbb{Z}\}$.*

Proposition 6.1. *Let $\mathbb{Y}$ be a closed proper subspace of a Banach space $\mathbb{X}$. Then $\mathbb{Y}$ is anti-coproximinal in $\mathbb{X}$ if $\overline{span\ \mathcal{J}_\mathbb{Y}}^{w^*} = \mathbb{X}^*$.*
Moreover, when $Sm(\mathbb{X}) \cap \mathbb{Y}$ is dense in $\mathbb{Y}$, then $\mathbb{Y}$ is anti-coproximinal in $\mathbb{X}$ if and only if $\overline{span\mathcal{J}_\mathbb{Y}}^{w^*} = \mathbb{X}^*$.

Proof. Suppose on the contrary that $\mathbb{Y}$ is not anti-coproximinal in $\mathbb{X}$. Then there exists an element $x \in \mathbb{X} \setminus \mathbb{Y}$ such that $\mathbb{Y} \perp_B x$. Let $x^* \in \mathbb{X}^*$. Since $\overline{span\ \mathcal{J}_\mathbb{Y}}^{w^*} = \mathbb{X}^*$, it follows that there exists a net $\{x^*_\alpha\}_{\alpha \in \Lambda} \in span\ \mathcal{J}_\mathbb{Y}$ such that $x^*_\alpha \xrightarrow{w^*} x^*$. Since $\mathbb{Y} \perp_B x$, for any $y \in Sm(\mathbb{X}) \cap \mathbb{Y}$, we have $y \perp_B x$. Using Lemma 6.1, we obtain that for any $y^* \in \mathcal{J}_\mathbb{Y}$, $y^*(x) = 0$ and therefore, $x^*_\alpha(x) = 0$, for each $\alpha \in \Lambda$. So, $x^*(x) = 0$. As x^* is taken arbitrarily from $\mathbb{X}^*$, we obtain that $x = 0$, a contradiction.
Let us now assume that $Sm(\mathbb{X}) \cap \mathbb{Y}$ is dense in $\mathbb{Y}$. We only need to prove the necessary part. Suppose on the contrary that $\overline{span\mathcal{J}_\mathbb{Y}}^{w^*} \subsetneq \mathbb{X}^*$. Clearly, ${}^\perp span\mathcal{J}_\mathbb{Y} = \cap_{y^* \in \mathcal{J}_\mathbb{Y}} \ker y^*$. Therefore, following Lemma 6.3, we conclude that $(\cap_{y^* \in \mathcal{J}_\mathbb{Y}} \ker y^*)^\perp = \overline{span\mathcal{J}_\mathbb{Y}}^{w^*}$. Thus,

$$(\cap_{y^* \in \mathcal{J}_\mathbb{Y}} \ker y^*)^* = \mathbb{X}^* / \overline{span\mathcal{J}_\mathbb{Y}}^{w^*},$$

which implies that $\cap_{y^* \in \mathcal{J}_\mathbb{Y}} \ker y^* \neq 0$. Let $z \in \cap_{y^* \in \mathcal{J}_\mathbb{Y}} \ker y^*$ and let $y \in \mathbb{Y}$ be arbitrary. Since $Sm(\mathbb{X}) \cap \mathbb{Y}$ is dense in $\mathbb{Y}$, there exists a sequence $\{y_n\} \subset Sm(\mathbb{X}) \cap \mathbb{Y}$ such that $y_n \longrightarrow y$. Note that for each $n \in \mathbb{N}$, $y_n^* \in \mathcal{J}_\mathbb{Y}$, where $J(y_n) = \{y_n^*\}$. As $z \in \cap_{y^* \in \mathcal{J}_\mathbb{Y}} \ker y^*$, we have $y_n \perp_B z$, for each $n \in \mathbb{N}$. Since $y_n \longrightarrow y$, it follows that $y \perp_B z$. This implies that $\mathbb{Y} \perp_B z$, contradicting the hypothesis that $\mathbb{Y}$ is anti-coproximinal in $\mathbb{X}$. □

We now give an example which illustrate the applicability of Proposition 6.1.

Example 6.1. *Let us consider the space* ℓ_p*, where* $p \in (1, \infty) \setminus \{2\}$ *and* $n \geq 3$*. Let* $e_i = (0, 0, \ldots, 0, \underset{i-th\ position}{1}, 0, \ldots)\ \forall i \in \mathbb{N}$*. Let* ϕ *be the canonical isometric isomorphism from* $(\ell_p)^*$ *to* ℓ_q*. We denote* $e_i' \in (\ell_p)^*$ *as* $e_i'(x_1, x_2, \ldots) = x_i\ \forall i \in \mathbb{N}$*. Suppose that* $\mathbb{Y} = span\{\widetilde{x}_1, \widetilde{x}_2, e_3, e_4, \ldots\}$*,* $\widetilde{x}_1 = (1, 1, 1, 0, 0, \ldots), \widetilde{x}_2 = (1, 2, 3, 0, 0, \ldots)$*. Let* $J(\widetilde{x}_i) = \{\widetilde{f}_i\}$*, for* $i = 1, 2$*. It is straightforward computation to verify that*

$$\phi(\widetilde{f}_1) = \Big(\frac{1}{3^{1-\frac{1}{p}}}, \frac{1}{3^{1-\frac{1}{p}}}, \frac{1}{3^{1-\frac{1}{p}}}, 0, 0, \ldots\Big) \in \ell_q,$$

$$\phi(\widetilde{f}_2) = \Big(\frac{1}{(1+2^p+3^p)^{1-\frac{1}{p}}}, \frac{2^{p-1}}{(1+2^p+3^p)^{1-\frac{1}{p}}}, \frac{3^{p-1}}{(1+2^p+3^p)^{1-\frac{1}{p}}}, 0, 0, \ldots\Big) \in \ell_q.$$

Consider the element $\widetilde{x} = 3\widetilde{x}_1 - \widetilde{x}_2 \in \mathbb{Y}$ *and let* $J(\widetilde{x}) = \{\widetilde{f}\}$*. Again we get*

$$\phi(\widetilde{f}) = \Big(\frac{2^{p-1}}{(2^p+1)^{1-\frac{1}{p}}}, \frac{1}{(2^p+1)^{1-\frac{1}{p}}}, 0, 0, \ldots\Big) \in \ell_q.$$

Observe that $\{\phi(\widetilde{f}_1), \phi(\widetilde{f}_2), \phi(\widetilde{f})\}$ *is a linearly independent set in* ℓ_q*. Since* $\phi(\widetilde{f}_1), \phi(\widetilde{f}_2), \phi(\widetilde{f}) \in \mathcal{J}_{\mathbb{Y}}$*, it follows that* $e_1', e_2', e_3' \in span\ \mathcal{J}_{\mathbb{Y}}$*. Moreover, as for any* $k > 3$*,* $e_k \in \mathbb{Y}$ *and* $J(e_k) = \{e_k'\}$*, we have* $e_k' \in \mathcal{J}_{\mathbb{Y}}$*. Therefore, for any* $k \in \mathbb{N}$*,* $e_k' \in span\ \mathcal{J}_{\mathbb{Y}}$*. This implies span* $\{e_k' : k \in \mathbb{N}\} \subset span\ \mathcal{J}_{\mathbb{Y}}$*. Therefore,*

$$(\ell_p)^* = \overline{span\ \{e_k' : k \in \mathbb{N}\}} \subseteq \overline{span\ \{e_k' : k \in \mathbb{N}\}}^{w^*} \subseteq \overline{span\ \mathcal{J}_{\mathbb{Y}}}^{w^*} \subseteq (\ell_p)^*.$$

So, $\overline{span\ \mathcal{J}_{\mathbb{Y}}}^{w^*} = (\ell_p)^*$ *and applying Theorem 6.1,* $\mathbb{Y}$ *is an anti-coproximinal subspace in* $\mathbb{X}$*.*

One of the main aims of this chapter is to illustrate the geometric specialty of strongly anti-coproximinal subspaces of a Banach space. We show this by deriving a necessary condition for a subspace to be strongly anti-coproximinal.

Theorem 6.2. *Let* $\mathbb{Y}$ *be a closed proper subspace of a Banach space* $\mathbb{X}$*. Suppose that* $S_{\mathbb{X}} \setminus S_{\mathbb{Y}}$ *contains an w-ALUR point. Then* $\mathbb{Y}$ *is not strongly anti-coproximinal in* $\mathbb{X}$*.*

Proof. Let $x \in S_{\mathbb{X}} \setminus S_{\mathbb{Y}}$ be an w-ALUR point. Suppose on the contrary that $\mathbb{Y}$ is strongly anti-coproximinal in $\mathbb{X}$. Thus for any $\epsilon \in [0, 1)$, $\mathbb{Y} \not\perp_B^{\epsilon} x$. Let us take $\{\epsilon_n\}_{n \in \mathbb{N}} \subset [0, 1)$ such that $\epsilon_n \longrightarrow 1$. For each $n \in \mathbb{N}$, there exists $y_n \in S_{\mathbb{Y}}$ such that $y_n \not\perp_B^{\epsilon_n} x$. Let $y_n^* \in S_{\mathbb{X}^*}$ be such that $y_n^*(y_n) = 1$. Since $B_{\mathbb{X}^*}$ is weak*-compact, it follows that there exists a weak*-cluster point $y^* \in B_{\mathbb{X}^*}$ of the sequence $\{y_n^*\}_{n \in \mathbb{N}}$. This implies $y^*(x)$ is a cluster point of $\{y_n^*(x)\}$. Following [22, Th. 8 (p. 72)], there exists a subsequence $\{y^*_{n_k}(x)\}$ of $\{y^*_n(x)\}$ such that $y^*_{n_k}(x) \to y^*(x)$. Now from Lemma 6.2, $y_{n_k} \not\perp_B^{\epsilon_{n_k}} x$ implies that $|y^*_{n_k}(x)| \geq \epsilon_{n_k}\|x\|$, for each $k \in \mathbb{N}$. So, $|y^*(x)| \geq \|x\|$, which in turn implies that $|y^*(x)| = \|x\|$. This implies $y^* \in J(x)$ or $y^* \in J(-x)$. Suppose that $y^* \in J(x)$. Note that

$$y^*_{n_k}\left(\frac{y_{n_k}+x}{2}\right) = \frac{y^*_{n_k}(y_{n_k})}{2} + \frac{y^*_{n_k}(x)}{2} = \frac{1}{2} + \frac{1}{2}y^*_{n_k}(x),$$

and so $\lim y^*_{n_k}(\frac{y_{n_k}+x}{2}) = 1$. Also, x being an w-ALUR point, we get that $y_{n_k} \xrightarrow{w} x$. As $\mathbb{Y}$ is a closed subspace of $\mathbb{X}$, so $x \in \mathbb{Y}$. This contradicts the fact that $x \in S_{\mathbb{X}} \setminus S_{\mathbb{Y}}$. If $y^* \in J(-x)$ then proceeding similarly we get $\lim y^*(\frac{y_{n_k}-x}{2}) = 1$ and consequently, $-x \in \mathbb{Y}$, again a contradiction. Thus $\mathbb{Y}$ is strongly anti-coproximinal in $\mathbb{X}$. □

Remark 6.3. *For a strictly convex Banach space $\mathbb{X}$, each $x \in S_{\mathbb{X}}$ is a w-ALUR point. Therefore, applying Theorem 6.2, we see that there does not exist any closed strongly anti-coproximinal subspace in $\mathbb{X}$. In particular, the above result tells that if the set of all w-ALUR points of $\mathbb{X}$ is nonempty, then a closed strongly-anticoproximinal subspace $\mathbb{Y}$ of $\mathbb{X}$ contains all w-ALUR points of $\mathbb{X}$. On the other hand, if $\mathbb{X}$ has no w-ALUR point then it might contain strongly anti-coproximinal subspaces. For an example consider $\mathbb{X} = \ell^3_\infty$. Following [54, Th. 2.25], it is strightforward to see that the subspace $\mathbb{Y} = span\{(3,0,2),(0,3,2)\}$ is strongly anti-coproximinal in $\mathbb{X}$.*

The following corollary is immediate from Theorem 6.2.

Corollary 6.1. *Let $\mathbb{Y}$ be a closed proper subspace of a Banach space $\mathbb{X}$. Suppose that the set of all w-ALUR points of $B_{\mathbb{X}}$ separates $\mathbb{X}^*$. Then $\mathbb{Y}$ is not strongly anti-coproximinal in $\mathbb{X}$.*

Proof. Consider $x^* \in \mathbb{X}^*$ such that $\mathbb{Y} \subseteq \ker x^*$. Suppose that the set of all w-ALUR points of $B_{\mathbb{X}}$ separates $\mathbb{X}^*$. Therefore, there exists a w-ALUR point $x \in S_{\mathbb{X}}$ such that $x^*(x) \neq 0$. Since $\mathbb{Y} \subseteq \ker x^*$, it follows that $x \notin \mathbb{Y}$. Thus $x \in S_{\mathbb{X}} \setminus S_{\mathbb{Y}}$ is a w-ALUR point and therefore, applying Theorem 6.2, we get the desired result. □

In the next theorem, we give another necessary condition for strongly anti-coproximinal finite-dimensional subspaces, further illustrating its geometric specialty.

Theorem 6.4. *Let $\mathbb{Y}$ be a strongly anti-coproximinal finite-dimensional subspace of a Banach space $\mathbb{X}$. Suppose that F is a maximal face of $B_{\mathbb{X}}$ and $x \in int(F)$. Then there exists $y \in \mathbb{Y}$ such that $J(x) \cap J(y) \neq \emptyset$.*

Proof. As $\mathbb{Y}$ is strongly anti-coproximinal in $\mathbb{X}$, we have $\mathbb{Y} \not\perp_B^{\epsilon} x$, for all $\epsilon \in [0,1)$. Let us take $\{\epsilon_n\}_{n\in\mathbb{N}} \subset [0,1)$ such that $\epsilon_n \longrightarrow 1$. For each $n \in \mathbb{N}$, there exists $y_n \in S_{\mathbb{Y}}$ such that $y_n \not\perp_B^{\epsilon_n} x$.

Let $y_n^* \in S_{\mathbb{X}^*}$ be such that $y_n^*(y_n) = 1$. Since $B_{\mathbb{X}^*}$ is weak*-compact, it follows that there exists a weak*-cluster point $y^* \in B_{\mathbb{X}^*}$ of the sequence $\{y_n^*\}_{n\in\mathbb{N}}$. This implies $y^*(x)$ is a cluster point of $\{y_n^*(x)\}$. Following [22, Th. 8 (p. 72)], there exists a subsequence $\{y_{n_k}^*(x)\}$ of $\{y_n^*(x)\}$ such that $y_{n_k}^*(x) \to y^*(x)$. Now from Lemma 6.2, $y_{n_k} \not\perp_B^{\epsilon_{n_k}} x$ implies that $|y_{n_k}^*(x)| \geq \epsilon_{n_k}\|x\|$, for each $k \in \mathbb{N}$. So, $|y^*(x)| \geq \|x\|$, which in turn implies that $|y^*(x)| = \|x\|$. This implies $y^* \in J(x)$ or $y^* \in J(-x)$. Let $y^* \in J(x)$. Since $\mathbb{Y}$ is finite-dimensional, it follows that $S_{\mathbb{Y}}$ is compact and therefore, $y_{n_k} \longrightarrow y$, for some $y \in S_{\mathbb{Y}}$. So for each $k \in \mathbb{N}$,

$$\begin{aligned} |y_{n_k}^*(y_{n_k}) - y^*(y)| &= |y_{n_k}^*(y_{n_k}) - y_{n_k}^*(y) + y_{n_k}^*(y) - y^*(y)| \\ &\leq \|y_{n_k}^*\|\|y_{n_k} - y\| + |y_{n_k}^*(y) - y^*(y)|. \end{aligned}$$

Taking $k \longrightarrow \infty$ in the above relation we get $y_{n_k}^*(y_{n_k}) \longrightarrow y^*(y)$. Since for each k, $y_{n_k}^*(y_{n_k}) = 1$, we have $y^*(y) = 1$. In other words, $y^* \in J(y)$. Thus we get $y^* \in J(x) \cap J(y)$. On the other hand, if $y^* \in J(-x)$ then $-y^* \in J(x)$. Then proceeding similarly as above, we can conclude that $J(x) \cap J(-y) \neq \emptyset$. This completes the proof.

□

The following observation, also geometric in nature, discusses the intersection property of finite-dimensional strongly anti-coproximinal subspaces of a Banach space.

Theorem 6.5. *Let $\mathbb{X}$ be a Banach space and let $\mathbb{Y}$ be a finite-dimensional subspace of $\mathbb{X}$. If $\mathbb{Y}$ is strongly anti-coproximinal in $\mathbb{X}$, then $\mathbb{Y}$ intersects every maximal face of $B_{\mathbb{X}}$.*

Proof. Let F be a maximal face of $B_{\mathbb{X}}$ and let $x \in int(F)$. From Theorem 6.4, we get an element $y \in \mathbb{Y}$ such that $J(x) \cap J(y) \neq \emptyset$. Suppose that $z^* \in J(x) \cap J(y)$, for some $y \in \mathbb{Y}$. As $x \in int(F)$ and $z^*(x) = 1$, it is easy to check that for any $v \in F$, $z^*(v) = 1$. Let $w \in co(F \cup \{y\})$. Then $w = (1-t)z + ty$, for some $t \in [0,1]$ and $z \in F$. The relation $z^*(y) = z^*(z) = 1$ yields that $z^*(w) = 1$. This implies that $w \in S_{\mathbb{X}}$. So, $co(F \cup \{y\}) \subset S_{\mathbb{X}}$. Therefore, there exists some face F' of $B_{\mathbb{X}}$ such that $F \subset co(F \cup \{y\}) \subset F'$. As F is a maximal face, we have $F = co(F \cup \{y\})$, so $y \in F$. Therefore, $\mathbb{Y} \cap F \neq \emptyset$, as desired. □

Remark 6.6. *It is easy to verify that whenever $x \in S_{\mathbb{X}}$ is a w-ALUR, $\{x\}$ is a maximal face. Following Theorem 6.2, for a closed proper subspace $\mathbb{Y}$ to be strongly anti-coproximinal in $\mathbb{X}$, $\mathbb{Y}$ has to intersect every maximal face of the form $\{x\}$, where $x \in S_{\mathbb{X}}$ is w-ALUR. On the other hand, Theorem 6.5 implies that a finite-dimensional strongly anti-coproximinal subspace of $\mathbb{X}$ intersects every maximal face of $B_{\mathbb{X}}$.*

Applying Theorem 6.5, we give an example of a Banach space having no strongly anti-coproximinal subspace.

Example 6.7. *Let $\mathbb{X}$ be the 3-dimensional Banach space whose unit ball is a prism-pyramid with hexagon base such that* $Ext(B_{\mathbb{X}}) = \{\pm(1,0,1), \pm(\frac{1}{2}, \frac{\sqrt{3}}{2}, 1), \pm(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 1), \pm(-1,0,1), \pm(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1), \pm(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1), \pm(0,0,2)\}$ *(see [48, Th. 2.5 (fig. 3)]). From the structure of the unit ball of $\mathbb{X}$, it can be observed that there does not exist any 2-dimensional subspace of $\mathbb{X}$ which intersects all the maximal faces of $B_{\mathbb{X}}$. Therefore, applying Theorem 6.5, there exists no strongly anti-coproximinal subspace in $\mathbb{X}$. Note that $\mathbb{X}$ has no w-ALUR points. Therefore, we are unable to apply Theorem 6.2 in this case. However, Theorem 6.5 ensures that there exists no strongly anti-coproximinal subspace of $\mathbb{X}$.*

As a consequence of Theorem 6.5, the following corollary is immediate.

Corollary 6.2. *Let $\mathbb{Y}$ be a finite-dimensional proper subspace of $\mathbb{X}$. Suppose that the set of all rotund points of unit ball of $B_{\mathbb{X}}$ separates $\mathbb{X}^*$. Then $\mathbb{Y}$ is not a strongly anti-coproximinal subspace of $\mathbb{X}$.*

For a finite-dimensional subspace $\mathbb{Y}$ of $\mathbb{X}$, if $S_{\mathbb{X}} \setminus S_{\mathbb{Y}}$ contains a rotund point then $\mathbb{Y}$ is not strongly anti-coproximinal subspace in $\mathbb{X}$.

Our next goal is to show that the only real Banach spaces which contain a finite-dimensional strongly anti-coproximinal subspace must be themselves finite-dimensional polyhedral. We require the following lemma for our purpose.

Lemma 6.4. *[48, Lemma 2.1] Let $\mathbb{X}$ be an n-dimensional polyhedral Banach space. Then $f \in S_{\mathbb{X}^*}$ is an extreme point of $B_{\mathbb{X}^*}$ if and only if f is a supporting functional corresponding to a maximal face of $B_{\mathbb{X}}$.*

Theorem 6.8. *Let $\mathbb{Y}$ be a finite-dimensional polyhedral subspace of a real Banach space $\mathbb{X}$. Suppose $\mathbb{Y}$ is strongly anti-coproximinal in $\mathbb{X}$. Then*

(i) for any two distinct maximal faces F_1, F_2 of $B_{\mathbb{X}}$, $F_1 \cap \mathbb{Y} \neq F_2 \cap \mathbb{Y}$

(ii) $\mathbb{X}$ is a finite-dimensional polyhedral Banach space.

Proof. (i) If F is a face of $B_{\mathbb{X}}$, then it clear that $F \cap B_{\mathbb{Y}}$ is also a face. Suppose on the contrary that F_1, F_2 are two distinct maximal faces of $B_{\mathbb{X}}$ such that $F_1 \cap \mathbb{Y} = F_2 \cap \mathbb{Y}$. Let $\pm Q_1, \pm Q_2, \ldots, \pm Q_r$ be the distinct maximal faces of $B_{\mathbb{Y}}$. Suppose that for each $1 \leq i \leq r$, $y_i^* \in S_{\mathbb{Y}^*}$ supports the face Q_i. Observe that this correspondence between y_i^* and Q_i is one-one (see Lemma 6.4). For each $1 \leq i \leq r$, suppose that $x_i^* \in S_{\mathbb{X}^*}$ is a Hahn-Banach extension of y_i^*. Take $x \in int(F_1)$. We claim that x_k^* supports the face F_1, for some $k, 1 \leq k \leq r$. Otherwise, let us assume that x_i^* do not support the face F_1, for any $1 \leq i \leq r$. Let $\max\{|x_i^*(x)| : 1 \leq i \leq r\} = \epsilon_0$. Clearly $\epsilon_0 < 1$. Let $y \in S_{\mathbb{Y}}$. Then $y \in Q_k$, for some $1 \leq k \leq r$. So, $x_k^* \in J(y)$ and $|x_k^*(x)| \leq \epsilon_0$.

Thus from Lemma 6.2, we get $y \perp_B^{\epsilon_0} x$. Therefore, $\mathbb{Y} \perp_B^{\epsilon_0} x$, a contradiction to the fact that $\mathbb{Y}$ is strongly anti-coproximinal in $\mathbb{X}$.This establishes our claim and so x_k^* supports the face F_1, for some $k, 1 \leq k \leq r$. As F_1 is a maximal face, observe that $Q_k \subset F_1$. Let $z^* \in S_{\mathbb{X}^*}$ support the face F_2. Clearly, $z^* \neq \pm x_k^*$. Let

$$\epsilon = \max\{|z^*(x)|, |x_i^*(x)| : i \in \{1, 2, \ldots, r\} \setminus \{k\}\}.$$

We claim that $\epsilon < 1$. First we note that $|z^*(x)| < 1$, as otherwise, $F_1 = F_2$. If possible, let $j \in \{1, 2, \ldots, r\} \setminus \{k\}$ be such that $x_j^*(x) = 1$. Then x_j^* supports F_1. It is clear that x_j^* and x_k^* support Q_j and Q_k, respectively. Since x_j^*, x_k^* support the maximal face F_1, it is easy to observe that $Q_j, Q_k \subset F_1 \cap B_{\mathbb{Y}}$. This contradicts Q_j, Q_k are two distinct maximal faces of $B_{\mathbb{Y}}$. Therefore, $\epsilon < 1$. Now take $y \in S_{\mathbb{Y}}$. If $y \in Q_j$, for some $j \in \{1, 2, \ldots, r\} \setminus \{k\}$ then $x_j^* \in J(y)$ and $|x_j^*(x)| \leq \epsilon$. Suppose that $y \in Q_k$. Since $Q_k \subset F_1 \cap \mathbb{Y} = F_2 \cap \mathbb{Y}$, it follows that $z^* \in J(y)$, and so, we again obtain that $|z^*(x)| \leq \epsilon$. This proves that $y \perp_B^{\epsilon} x$. Since $y \in S_{\mathbb{Y}}$ is arbitrary, $\mathbb{Y} \perp_B^{\epsilon} x$, contradicting the fact that $\mathbb{Y}$ is strongly anti-coproximinal in $\mathbb{X}$.

(ii) Suppose that $\pm Q_1, \pm Q_2, \ldots, \pm Q_r$ are the maximal faces of $B_{\mathbb{Y}}$. Clearly $Q_i = F_i \cap B_{\mathbb{Y}}$, for some maximal face F_i of $B_{\mathbb{X}}$. We claim that $\pm F_1, \pm F_2, \ldots, \pm F_r$ are the only maximal faces of $B_{\mathbb{X}}$. If possible, let F be a maximal face of $B_{\mathbb{X}}$ such that $F \neq \pm F_i$, for all $i = 1, 2, \ldots, r$. From Theorem 6.5, we note that $F \cap B_{\mathbb{Y}} \neq \emptyset$. Clearly, $F \cap B_{\mathbb{Y}}$ is a maximal face of $B_{\mathbb{Y}}$. Therefore, $F \cap B_{\mathbb{Y}} = Q_j = F_j \cap B_{\mathbb{Y}}$, for some $j \in \{1, 2, \ldots, r\}$. Since $F \neq \pm F_j$, following (i) we arrive at a contradiction. This proves our claim and consequently, $\mathbb{X}$ is a finite-dimensional polyhedral Banach space.

$\square$

Remark 6.9. *We observe that the condition given in Theorem 6.5 is necessary but not sufficient. Consider the space* $\mathbb{X} = \ell_1^3(\mathbb{R})$ *and the subspace* $\mathbb{Y} = span\{(1, 0, 0), (0, 1, 0)\}$*. Then it is easy to see that the subspace* $\mathbb{Y}$ *intersects all the maximal faces of* $B_{\mathbb{X}}$*, whereas* $\mathbb{Y}$ *is a coproximinal subspace. Indeed, given any* $(x, y, z) \in \mathbb{X} \setminus \mathbb{Y}$*,* $(x, y, 0) \in \mathbb{Y}$ *is a best coapproximation to* (x, y, z) *out of* $\mathbb{Y}$*. This shows that the subspace* $\mathbb{Y}$ *may intersect every maximal face of* $B_{\mathbb{X}}$ *but is not strongly anti-coproximinal in* $\mathbb{X}$*. However, as we observe in the next result, if* $\mathbb{Y}$ *intersects the relative interior of every maximal face of* $B_{\mathbb{X}}$*, then* $\mathbb{Y}$ *is indeed strongly anti-coproximinal in* $\mathbb{X}$*.*

Theorem 6.10. *Let* $\mathbb{X}$ *be a Banach space and let* $\mathbb{Y}$ *be a proper subspace of* $\mathbb{X}$*. If* $\mathbb{Y}$ *intersects the relative interior of every facet of* $B_{\mathbb{X}}$ *then* $\mathbb{Y}$ *is strongly anti-coproximinal in* $\mathbb{X}$*.*

Proof. Suppose on the contrary that $\mathbb{Y}$ is not strongly anti-coproximinal in $\mathbb{X}$. Then there exists

$x \in S_{\mathbb{X}} \setminus S_{\mathbb{Y}}$ and $\epsilon \in [0,1)$ such that $\mathbb{Y} \perp_B^{\epsilon} x$. Let us consider a maximal face $F \subset S_{\mathbb{X}}$ such that $x \in F$. Since $\mathbb{Y} \cap int(F) \neq \emptyset$, let us take $y \in S_{\mathbb{Y}} \cap int(F)$. Then there exists $z \in F$ such that $y = (1-t)x + tz$, for some $t \in (0,1)$. It is immediate that for any $y^* \in J(y)$, we get $y^*(x) = 1$. This implies $y \not\perp_B^{\epsilon} x$, a contradiction. This establishes the theorem. □

We end this section with the following remark.

Remark 6.11. *We would like to mention that we could not establish whether the above-mentioned sufficient condition for strongly anti-coproximinal subspaces is also necessary. However, in case of a finite-dimensional polyhedral Banach space $\mathbb{X}$, it is known that a subspace $\mathbb{Y}$ of $\mathbb{X}$ is strongly anti-coproximinal if and only if $\mathbb{Y}$ intersects the relative interior of every facet of $B_{\mathbb{X}}$ (see [54, Th. 2.20]).*

6.3 Anti-coproximinality in the space of bounded linear operators

In this section, we study the anti-coproximinality (strongly anti-coproximinality) in the space of all bounded linear operators defined on Banach spaces using the well known *Bhatia-Šemrl property*. Before proceeding further first let us recall the characterization of Birkhoff-James orthogonality in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. As $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ can be embedded into $\ell_\infty(S_{\mathbb{X}}, \mathbb{Y})$, using Theorem 5.5 we obtain the following characterization.

Theorem 6.12. *Let $\mathbb{X}, \mathbb{Y}$ be two Banach spaces and let $T, A \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$. Suppose that $\epsilon \in [0,1)$. Then $T \perp_B^{\epsilon} A$ if and only if*

$$co\Bigg(\bigg\{ \lim y_n^*(Ax_n) : (x_n, y_n^*) \in S_{\mathbb{X}} \times S_{\mathbb{Y}^*}, \lim y_n^*(Tx_n) = 1 \bigg\}\Bigg) \cap \mathcal{D}(\epsilon) \neq \emptyset.$$

6.3.1 Bhatia-Šemrl (BŠ) Property

Recall that an operator $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ satisfies *Bhatia-Šemrl (BŠ) Property* [49] if for any $A \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, $T \perp_B A$ if and only if $Tx \perp_B Ax$, for some $x \in M_T$. Observe that, in general, *Bhatia-Šemrl property* does not hold for an arbitrary bounded linear operator defined on a Banach space. However the same holds under additional restrictions on the space as well as the operator. For more information on *BŠ* property, readers may go through [23, 49, 51] and the references therein. We show that if $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ is an absolutely strongly exposing operator then T satisfies BŠ property. Note that an operator $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ is said to be an absolutely strongly exposing operator (see [21]) if there exists $x_0 \in B_{\mathbb{X}}$ such that whenever a sequence $\{x_n\}$ in $B_{\mathbb{X}}$ satisfies $\|Tx_n\| \to \|T\|$, then there exists a sequence $\{\theta_n\}$ of elements of $\mathbb{T}$ such that $\theta_n x_n \to x_0$. It is easy to observe that for such an operator T, $M_T = \{\mu x_0 : |\mu| = 1\}$, for some $x_0 \in S_{\mathbb{X}}$. The set of absolutely strongly exposing operators of $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ is denoted by $ASE(\mathbb{X}, \mathbb{Y})$.

Theorem 6.13. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces and let $\epsilon \in [0, 1)$. Suppose that $T \in ASE(\mathbb{X}, \mathbb{Y})$ with $M_T = \{\mu x_0 : |\mu| = 1\}$. Then for any $A \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ the following are equivalent:*

(i) $T \perp_B^{\epsilon} A$

(ii) $\Omega \cap \mathcal{D}(\epsilon) \neq \emptyset$, *where* $\Omega = \{y^*(Ax_0) : \ y^* \in J(Tx_0)\}$

Moreover, if $M_T \subseteq M_A$, then $T \perp_B^{\epsilon} A$ if and only if $Tx_0 \perp_B^{\epsilon} Ax_0$.

Proof. Following Theorem 6.12, $T \perp_B^{\epsilon} A$ if and only if $co(\Omega') \cap \mathcal{D}(\epsilon) \neq \emptyset$, where

$$\Omega' = \left\{ \lim y_n^*(Ax_n) : (x_n, y_n^*) \in S_{\mathbb{X}} \times S_{\mathbb{Y}^*}, \ \lim y_n^*(Tx_n) = \|T\| \right\}.$$

Clearly, $\Omega \subseteq \Omega'$. Since $J(Tx_0)$ is convex, it is easy to see that Ω is convex. Therefore, to complete the theorem, we only need to show $\Omega' \subseteq \Omega$. Suppose that $\lambda \in \Omega'$. Then there exists $(x_n, y_n^*) \in S_{\mathbb{X}} \times S_{\mathbb{Y}^*}$ such that $y_n^*(Tx_n) \to \|T\|$ and $y_n^*(Ax_n) \to \lambda$. Without loss of generality we may assume that $\|Tx_n\| \to c$, a real number. As

$$\|T\| = \lim y_n^*(Tx_n) \leq \lim \|Tx_n\| = c \leq \|T\|,$$

it is clear that $\|Tx_n\| \to \|T\|$. Now, since $T \in ASE(\mathbb{X}, \mathbb{Y})$, there exists $x \in B_{\mathbb{X}}$ and $\{\theta_n\} \subset \mathbb{T}$ such that $\theta_n x_n \to x$. Clearly, $x = \mu x_0$, for some $\mu \in \mathbb{T}$. Again, passing onto a subsequence, if necessary, we may assume that $\theta_n \to \theta \in \mathbb{T}$. Then $x_n \to \mu\bar{\theta}x_0$ and so $Tx_n \to \mu\bar{\theta}Tx_0$. Since $B_{\mathbb{Y}^*}$ is weak*-compact, it follows that there exists $y^* \in B_{\mathbb{Y}^*}$ such that $y_n^* \xrightarrow{w^*} y^*$. Thus $y_n^*(Tx_n) \to y^*(\mu\bar{\theta}Tx_0)$. Then $\mu\bar{\theta}y^*(Tx_0) = \lim y_n^*(Tx_n) = \|T\|$ and so $\mu\bar{\theta}y^* \in J(Tx_0)$. Also, $y_n^*(Ax_n) \to \mu\bar{\theta}y^*(Ax_0)$. This implies $\mu\bar{\theta}y^*(Ax_0) = \lim y_n^*(Ax_n) = \lambda$. Therefore, $\lambda \in \Omega$.

Let $M_T \subseteq M_A$. So, $\|Ax_0\| = \|Tx_0\| = 1$. Since $T \perp_B^{\epsilon} A$, it follows that (ii) holds. Let $\rho \in \Omega \cap \mathcal{D}(\epsilon)$. Suppose that $\rho = y^*(Ax_0)$, for some $y^* \in J(Tx_0)$. As $|\rho| \leq \epsilon = \epsilon\|Ax_0\|$, applying Lemma 2.2 we obtain that $Tx_0 \perp_B^{\epsilon} Ax_0$. □

In this context, we note that in [34, Th. 3.2], considering $\mathbb{X}$ as reflexive and $T, A \in \mathbb{K}(\mathbb{X}, \mathbb{Y})$, it was shown that whenever $M_T \subseteq M_A, T \perp_B^{\epsilon} A$ if and only if there exists $x \in M_T$ such that $Tx \perp_B^{\epsilon} Ax$.

Theorem 6.14. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces. Suppose that $T \in ASE(\mathbb{X}, \mathbb{Y})$ with $M_T = \{\mu x_0 : |\mu| = 1\}$. Then for any $A \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$, $T \perp_B A$ if and only if $Tx_0 \perp_B Ax_0$.*

Proof. We only prove the necessary part as the sufficient part follows easily. Following Theorem 6.13, $T \perp_B A$ if and only if $0 \in \Omega$, where $\Omega = \{y^*(Ax_0) : y^* \in J(Tx_0)\}$. Applying Lemma 6.1, we get $Tx_0 \perp_B Ax_0$, as desired.

□

In [23, Th. 2.4], it was proved that *"For a real Banach space* $\mathbb{X}$*, if the closed unit ball* $B_{\mathbb{X}}$ *is an RNP set, then the set of norm attaining operators satisfying the BŠ Property is dense in* $\mathbb{L}(\mathbb{X}, \mathbb{Y})$*, for every Banach space* $\mathbb{Y}$*"*. In the following result, we prove this result for both real and complex Banach spaces.

Corollary 6.3. *Let* $\mathbb{X}$ *be a Banach space with Radon-Nikodym Property and let* $\mathbb{Y}$ *be any Banach space. The set of norm attaining operators satisfying* BŠ-Property *is dense in* $\mathbb{L}(\mathbb{X}, \mathbb{Y})$.

Proof. Since $\mathbb{X}$ satisfies Radon-Nikodym Property, it follows from [3, Th. 5] that for any given Banach space $\mathbb{Y}$, $ASE(\mathbb{X}, \mathbb{Y})$ is dense in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. Following Theorem 6.14, when $T \in ASE(\mathbb{X}, \mathbb{Y})$, we have that T satisfies *BŠ property*, thus completing the proof. □

6.3.2 Anti-coproximinal subspaces in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$

We next study the subspace $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ of $\mathbb{L}(\mathbb{X}, \mathbb{Y})$, from the perspective of best coapproximation.

Theorem 6.15. *Let* $\mathbb{X}$ *be a Banach space such that the set of all strongly exposed points of* $B_{\mathbb{X}}$ *separates* $\mathbb{X}^*$*. Let* $\mathbb{Y}$ *be any Banach space. Then exactly one of the following holds true:*

(i) $\mathbb{K}(\mathbb{X}, \mathbb{Y}) = \mathbb{L}(\mathbb{X}, \mathbb{Y})$.

(ii) $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ *is anti-coproximinal in* $\mathbb{L}(\mathbb{X}, \mathbb{Y})$.

Proof. Let us assume that $\mathbb{K}(\mathbb{X}, \mathbb{Y}) \subsetneq \mathbb{L}(\mathbb{X}, \mathbb{Y})$. Suppose on the contrary that $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ is not anti-coproximinal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. Then there exists $S \in \mathbb{L}(\mathbb{X}, \mathbb{Y}) \setminus \mathbb{K}(\mathbb{X}, \mathbb{Y})$ such that

$$\mathbb{K}(\mathbb{X}, \mathbb{Y}) \perp_B S.$$

Without loss of generality, assume that $\|S\| = 1$. Since *st-Exp*$(B_{\mathbb{X}})$ separates $\mathbb{X}^*$ it is straightforward to see that there exists $z \in$ *st-Exp*$(B_{\mathbb{X}})$ such that $Sz \neq 0$. Suppose that $Sz = w$ and $z^* \in J(z)$ such that $M_{z^*} = \{\mu z : |\mu| = 1\}$. Define $T : \mathbb{X} \longrightarrow \mathbb{Y}$ by

$$Tx = z^*(x)w, \text{ for any } x \in \mathbb{X}.$$

Note that for any $x \in S_{\mathbb{X}}$, $\|Tx\| = |z^*(x)|\|w\| \leq \|w\|$. As $\|Tz\| = |z^*(z)|\|w\| = \|w\| > |z^*(x)|\|w\| = \|Tx\|$, for all $x \in S_{\mathbb{X}} \setminus \{\mu z : |\mu| = 1\}$, so $\|T\| = \|w\|$ and $M_T = \{\mu z : |\mu| = 1\}$.

Clearly, $T \in \mathbb{K}(\mathbb{X}, \mathbb{Y})$. On the other hand, let $\{x_n\} \subset S_{\mathbb{X}}$ be such that $\|Tx_n\| \longrightarrow \|T\| = \|w\|$. Therefore, $\lim \|z^*(x_n)w\| = \|T\| = \|w\|$. This implies $\lim |z^*(x_n)| = 1$. Therefore, there exists a sequence $\{\theta_n\} \subset \mathbb{T}$ such that $\theta_n z^*(x_n) \longrightarrow 1$, which implies that $z^*(\theta_n x_n) \longrightarrow 1 = z^*(z)$. Since z is strongly exposed point of $B_{\mathbb{X}}$, we have $\theta_n x_n \longrightarrow z$. Therefore $T \in ASE(\mathbb{X}, \mathbb{Y})$. Since $\mathbb{K}(\mathbb{X}, \mathbb{Y}) \perp_B S$ we have $T \perp_B S$. As $M_T = \{\mu z : |\mu| = 1\}$, applying Theorem 6.14, we obtain that $Tz \perp_B Sz$, a contradiction. This completes the proof of the theorem. □

Remark 6.16. *It is worth mentioning here that there does not exist any best coapproximation to the identity operator I out of the subspace $\mathbb{K}(\mathbb{X})$, whenever $\mathbb{X}$ is an infinite-dimensional Banach space and $\mathbb{K}(\mathbb{X})$ is semi-M-ideal in span$\{I, \mathbb{K}(\mathbb{X})\}$, (see [39, Th. 14]).*

Theorem 6.17. *Let $\mathbb{X}$ be a Banach space such that st-Exp$(B_{\mathbb{X}})$ separates $\mathbb{X}^*$ and $\mathbb{Y}$ be any Banach space. If $\mathbb{Z}$ is anti-coproximinal in $\mathbb{Y}$ then any subspace $\mathcal{W}$ of $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ containing $\mathcal{F}(\mathbb{X}, \mathbb{Z})$ is anti-coproximinal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$.*

Proof. Let $\mathcal{W} \perp_B A$. Take $\widetilde{x} \in$ *st-Exp*$(B_{\mathbb{X}})$ and let $f_{\widetilde{x}} \in B_{\mathbb{X}^*}$ strongly exposes the point $\widetilde{x}$. Let $z \in S_{\mathbb{Z}}$. Define $T_z : \mathbb{X} \to \mathbb{Y}$ be such that $T_z(x) = f_{\widetilde{x}}(x)z \ \forall x \in \mathbb{X}$. Clearly, $T_z \in \mathcal{F}(\mathbb{X}, \mathbb{Y}) \subset \mathcal{W}$ and $\|T_z\| = 1$. Observe that $M_{T_z} = \{\mu\widetilde{x} : |\mu| = 1\}$. Let $\{x_n\} \subset S_{\mathbb{X}}$ be such that $\|T_z(x_n)\| \longrightarrow \|T_z\|$. Therefore, $\lim \|f_{\widetilde{x}}(x_n)z\| = \|T_z\|$. This implies $\lim |f_{\widetilde{x}}(x_n)| = 1$. Therefore, there exists a sequence $\{\theta_n\} \subset \mathbb{T}$ such that $\theta_n f_{\widetilde{x}}(x_n) \longrightarrow 1$, which implies that $f_{\widetilde{x}}(\theta_n x_n) \longrightarrow 1 = f_{\widetilde{x}}(z)$. Since $\widetilde{x}$ is a strongly exposed point of $B_{\mathbb{X}}$, we have $\theta_n x_n \longrightarrow \widetilde{x}$. Therefore $T \in ASE(\mathbb{X}, \mathbb{Z})$. Hence for any $z \in S_{\mathbb{Z}}$, $T_z \perp_B A$, following Theorem 6.13, we get

$$T_z(\widetilde{x}) \perp_B A(\widetilde{x}) \implies z \perp_B A(\widetilde{x}) \implies \mathbb{Z} \perp_B A(\widetilde{x}).$$

As $\mathbb{Z}$ is anti-coproximinal in $\mathbb{X}$, we get $A(\widetilde{x}) = 0$. As *st-Exp*$(B_{\mathbb{X}})$ separates $\mathbb{X}^*$ and for any $x \in$ *st-Exp*$(B_{\mathbb{X}})$, $A(x) = 0$, we get $A = 0$. Therefore, $\mathcal{W}$ is anti-coproximinal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. □

In the above theorem if we consider an additional assumption on the subspace of $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ then we obtain a characterization of anti-coproximinality of those subspaces.

Theorem 6.18. *Let $\mathbb{X}, \mathbb{Y}$ be a Banach space such that st-Exp$(B_{\mathbb{X}})$ separates $\mathbb{X}^*$. Suppose that $\mathcal{W} \subset \mathbb{L}(\mathbb{X}, \mathbb{Y})$ is a subspace containing $\mathcal{F}(\mathbb{X}, \mathbb{Z})$ such that $ASE(\mathbb{X}, \mathbb{Z}) \cap \mathcal{W}$ is dense in $\mathcal{W}$. Then $\mathcal{W}$ is anti-coproximinal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ if and only if $\mathbb{Z}$ is anti-coproximinal in $\mathbb{Y}$.*

Proof. Since the sufficient part follows from Theorem 6.17, we only need to prove the necessary part. Suppose on the contrary that $\mathbb{Z}$ is not anti-coproximinal in $\mathbb{Y}$. This implies there exists $y_0 \in \mathbb{Y}$ such that $\mathbb{Z} \perp_B y_0$. Let $f \in S_{\mathbb{X}^*}$. Define $A : \mathbb{X} \to \mathbb{Y}$ such that $A(x) = f(x)y_0$. Let $T \in ASE(\mathbb{X}, \mathbb{Z}) \cap S_{\mathcal{W}}$ and let $M_T = \{\mu x_0 : \mu \in \mathbb{T}\}$, for some $x_0 \in S_{\mathbb{X}}$. As $\mathbb{Z} \perp_B y_0$, we get $Tx_0 \perp_B y_0 \implies Tx_0 \perp_B Ax_0$. Following Theorem 6.13, we get $T \perp_B A$. This implies $ASE(\mathbb{X}, \mathbb{Z}) \cap S_{\mathcal{W}} \perp_B A$. Since $ASE(\mathbb{X}, \mathbb{Z}) \cap \mathcal{W}$ is dense in $\mathcal{W}$, we get $\mathcal{W} \perp_B A$. This contradicts $\mathcal{W}$ is anti-coproximinal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. □

6.3.3 Strongly anti-coproximinal subspaces in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$

Theorem 6.19. *Let $\mathbb{X}, \mathbb{Y}$ be Banach spaces. Suppose that $B_{\mathbb{X}}$ is the closed convex hull of its strongly exposed points. Then exactly one of the following holds true:*

(i) $\mathbb{K}(\mathbb{X}, \mathbb{Y}) = \mathbb{L}(\mathbb{X}, \mathbb{Y})$.

(ii) $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ *is strongly anti-coproximinal in* $\mathbb{L}(\mathbb{X}, \mathbb{Y})$.

Proof. Let us assume that $\mathbb{K}(\mathbb{X}, \mathbb{Y}) \subsetneq \mathbb{L}(\mathbb{X}, \mathbb{Y})$. Suppose on the contrary that $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ is not strongly anti-coproximinal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. Then there exists $S \in \mathbb{L}(\mathbb{X}, \mathbb{Y}) \setminus \mathbb{K}(\mathbb{X}, \mathbb{Y})$ and $\epsilon \in [0, 1)$ such that

$$\mathbb{K}(\mathbb{X}, \mathbb{Y}) \perp_B^{\epsilon} S.$$

Without loss of generality we assume $\|S\| = 1$. Since $B_{\mathbb{X}}$ is the closed convex hull of its strongly exposed points, it follows that there exists $z \in S_{\mathbb{X}}$ such that $\|Sz\| > \epsilon$, where z is a strongly exposed point of $B_{\mathbb{X}}$. Otherwise, from convexity of norm we can write

$$\begin{aligned} \|S\| = \sup\{\|Sx\| : x \in B_{\mathbb{X}}\} &= \sup\{\|Sx\| : x \in \overline{co(st\text{-}Exp(B_{\mathbb{X}}))}\} \\ &= \sup\{\|Sx\| : x \in co(st\text{-}Exp(B_{\mathbb{X}}))\} \\ &= \sup\{\|Sx\| : x \in st\text{-}Exp(B_{\mathbb{X}})\} \leq \epsilon < 1. \end{aligned}$$

Suppose that $Sz = w$ and $z^* \in J(z)$ such that $M_{z^*} = \{\mu z : |\mu| = 1\}$. Now we define $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ such that $Tx = z^*(x)w$ for all $x \in \mathbb{X}$. Now proceeding similarly as in the proof of Theorem 6.15, we obtain that $T \in ASE(\mathbb{X}, \mathbb{Y})$. Clearly, $M_T = \{\mu z : |\mu| = 1\}$. Since $\mathbb{K}(\mathbb{X}, \mathbb{Y}) \perp_B^{\epsilon} S$ we have $T \perp_B^{\epsilon} S$. Following Theorem 6.13, $\Omega \cap \mathcal{D}(\epsilon) \neq \emptyset$, where $\Omega = \{y^*(Sz) : y^* \in J(Tz)\}$. Since $Tz = Sz = w$, it follows that $\Omega = \{\|w\|\}$. However, $\|w\| = \|Sz\| > \epsilon$, a contradiction. This establishes the theorem.

□

From [14, p. 121], we note that a Banach space $\mathbb{X}$ satisfies the Radon-Nikodym Property if and only if every bounded subset of $\mathbb{X}$ is dentable. Also, in [35, Th. 9] Phelps proved that every bounded subset of $\mathbb{X}$ is dentable if and only if every bounded closed convex subset of $\mathbb{X}$ is the closed convex hull of its strongly exposed points. Combining these results with Theorem 6.19, we obtain the following results.

Corollary 6.4. *Suppose that $\mathbb{X}$ is a Banach space with the Radon-Nikodym Property. Then either of the following holds true:*

(i) $\mathbb{K}(\mathbb{X}, \mathbb{Y}) = \mathbb{L}(\mathbb{X}, \mathbb{Y})$.

(ii) $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ *is strongly anti-coproximinal in* $\mathbb{L}(\mathbb{X}, \mathbb{Y})$.

Corollary 6.5. *Let $\mathbb{X}$ be a Banach space and let $\mathbb{Y}$ be a reflexive Banach space. Then either of the following holds true:*

(i) $\mathbb{K}(\mathbb{X}, \mathbb{Y}) = \mathbb{L}(\mathbb{X}, \mathbb{Y})$.

(ii) $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ *is strongly anti-coproximinal in* $\mathbb{L}(\mathbb{X}, \mathbb{Y})$.

Proof. Let us assume that $\mathbb{K}(\mathbb{X}, \mathbb{Y}) \subsetneqq \mathbb{L}(\mathbb{X}, \mathbb{Y})$. Suppose on the contrary that $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ is not strongly anti-coproximinal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. Then there exists $S \in \mathbb{L}(\mathbb{X}, \mathbb{Y}) \setminus \mathbb{K}(\mathbb{X}, \mathbb{Y})$ and $\epsilon \in [0, 1)$ such that $\mathbb{K}(\mathbb{X}, \mathbb{Y}) \perp_B^{\epsilon} S$. So, for any $T \in \mathbb{K}(\mathbb{X}, \mathbb{Y})$, $T \perp_B^{\epsilon} S$. It is straightforward to see that $T^* \perp_B^{\epsilon} S^*$. Since $\mathbb{Y}$ is reflexive, for any $A \in \mathbb{K}(\mathbb{Y}^*, \mathbb{X}^*)$ there exists $T \in \mathbb{K}(\mathbb{X}, \mathbb{Y})$ such that $T^* = A$. So, for any $A \in \mathbb{K}(\mathbb{Y}^*, \mathbb{X}^*)$, $A \perp_B^{\epsilon} S^*$. This implies $\mathbb{K}(\mathbb{Y}^*, \mathbb{X}^*) \perp_B^{\epsilon} S^*$. As $\mathbb{Y}$ is reflexive, applying Theorem 6.19, either $\mathbb{K}(\mathbb{Y}^*, \mathbb{X}^*) = \mathbb{L}(\mathbb{Y}^*, \mathbb{X}^*)$ or $\mathbb{K}(\mathbb{Y}^*, \mathbb{X}^*)$ is strongly anti-coproximinal in $\mathbb{L}(\mathbb{Y}^*, \mathbb{X}^*)$, both of which lead to a contradiction, this completes the proof. $\square$

Proposition 6.2. *Let $\mathbb{H}$ be a Hilbert space and let $\mathbb{Y}$ be a proper subspace of $\mathbb{L}(\mathbb{H})$. Suppose that for any $x, y \in S_{\mathbb{H}}$, there exists an $A \in \mathbb{Y}$ such that $Ax = y$ and $M_A = \{\pm x\}$. Then $\mathbb{Y}$ is strongly anti-coproximinal.*

Proof. Suppose on the contrary that $\mathbb{Y}$ is not strongly anti-coproximinal. Then there exists $\epsilon \in [0, 1), S \in \mathbb{L}(\mathbb{H}) \setminus \mathbb{Y}$ such that $\mathbb{Y} \perp_B^{\epsilon} S$. Let $x \in S_{\mathbb{H}}$. If $Sx \neq 0$, let $y = \frac{Sx}{\|Sx\|}$. Take $A \in \mathbb{Y}$ such that $Ax = y$ and $M_A = \{x\}$. Clearly, $\|A\| = 1$. Since $A \perp_B^{\epsilon} S$, applying [34, Th. 3.1],

$$|\langle Ax, Sx \rangle| \leq \epsilon \|S\| \|A\| \implies \|Sx\| \leq \epsilon \|S\|.$$

This is true for every $x \in S_{\mathbb{H}}$ such that $Sx \neq 0$. Observe that

$$\|S\| = \sup\{\|Sx\| : x \in S_{\mathbb{H}}\} \leq \epsilon \|S\| < \|S\|,$$

a contradiction. $\square$

Remark 6.20. *Let us consider the proper subspace $\mathbb{Y}$ of $\mathbb{L}(\mathbb{H})$, consisting of the finite-rank operators. For any $x, y \in S_{\mathbb{H}}$, consider an operator $A \in \mathbb{L}(\mathbb{H})$ such that $Ax = y$ and $Az = 0$, for all $z \in x^{\perp}$, where $x^{\perp} = \{z \in \mathbb{H} : x \perp_B z\}$. It is trivial to see that $A \in \mathbb{Y}$. Thus, from Proposition 6.2, we conclude that $\mathbb{Y}$ is a strongly anti-coproximinal subspace of $\mathbb{L}(\mathbb{H})$.*

We next provide a characterization of strongly anti-coproximinal subspaces in $\mathbb{L}(\mathbb{X},\mathbb{Y})$, whenever $\mathbb{X},\mathbb{Y}$ are finite-dimensional real polyhedral Banach spaces.

Proposition 6.3. *Let $\mathbb{X},\mathbb{Y}$ be two finite-dimensional real polyhedral Banach spaces. Then the following are equivalent:*

(i) $\mathbb{Z}$ is strongly anti-coproximinal in $\mathbb{L}(\mathbb{X},\mathbb{Y})$

(ii) For any $x \in Ext(B_{\mathbb{X}})$ and $y^* \in Ext(B_{\mathbb{Y}^*})$, there exists $A \in \mathbb{Z}$ such that $M_A = \{\pm x\}$ and $J(Ax) = \{y^*\}$.

Proof. As $\mathbb{X},\mathbb{Y}$ are finite-dimensional real polyhedral Banach spaces, $\mathbb{L}(\mathbb{X},\mathbb{Y})$ is also polyhedral. Moreover, from [42, Th. 1.3]

$$Ext(B_{\mathbb{L}(\mathbb{X},\mathbb{Y})^*}) = \{y^* \otimes x : x \in Ext(B_{\mathbb{X}}), y^* \in Ext(B_{\mathbb{Y}^*})\}.$$

Using [54, Th. 2.20], $\mathbb{Z}$ is strongly anti-coproximinal in $\mathbb{L}(\mathbb{X},\mathbb{Y})$ if and only if $\mathcal{J}_{\mathbb{Z}} = Ext(B_{\mathbb{L}(\mathbb{X},\mathbb{Y})^*})$. It is now straightforward to check that $y^* \otimes x \in \mathcal{J}_{\mathbb{Z}}$ if and only if there exists $A \in \mathbb{Z}$ such that $M_A = \{\pm x\}$ and $J(Ax) = \{y^*\}$. Hence we obtain the desired result. $\square$

Next we deal with the comparative study on strong anti-coproximinality between the subspaces of the operator space and subspaces of the target space. Then similarly as Theorem 6.18 we obtain results for strongly anti-coproximinal subspace of $\mathbb{L}(\mathbb{X},\mathbb{Y})$.

Theorem 6.21. *Let $\mathbb{X}$ be a Banach space such that $B_{\mathbb{X}}$ is the closed convex hull of its strongly exposed points and $\mathbb{Y}$ be any Banach space. If $\mathbb{Z}$ be strongly anti-coproximinal in $\mathbb{Y}$ then any subspace $\mathcal{W}$ of $\mathbb{L}(\mathbb{X},\mathbb{Y})$ containing $\mathcal{F}(\mathbb{X},\mathbb{Z})$ is strongly anti-coproximinal in $\mathbb{L}(\mathbb{X},\mathbb{Y})$.*

Proof. Let $\mathcal{W} \perp_B^{\epsilon} A$, for some $\epsilon \in [0,1)$. Take $\widetilde{x} \in st\text{-}Exp(B_{\mathbb{X}})$ and suppose $f_{\widetilde{x}} \in B_{\mathbb{X}^*}$ strongly exposes the point $\widetilde{x}$. Let $z \in S_{\mathbb{Z}}$. Define $T_z : \mathbb{X} \to \mathbb{Y}$ be such that $T_z(x) = f_{\widetilde{x}}(x)z \ \forall x \in \mathbb{X}$. Clearly, $T_z \in \mathcal{F}(\mathbb{X},\mathbb{Y}) \subset \mathcal{W}$ and $\|T_z\| = 1$. Following similar argument as in Theorem 6.17, $T \in ASE(\mathbb{X},\mathbb{Z})$ and $M_{T_z} = \{\mu\widetilde{x} : |\mu| = 1\}$. As for any $z \in S_{\mathbb{Z}}$, $T_z \perp_B^{\epsilon} A$, so following Theorem 6.13, we get

$$T_z(\widetilde{x}) \perp_B^{\epsilon} A(\widetilde{x}) \implies z \perp_B^{\epsilon} A(\widetilde{x}) \implies \mathbb{Z} \perp_B^{\epsilon} A(\widetilde{x}).$$

As $\mathbb{Z}$ is strongly anti-coproximinal in $\mathbb{X}$, we get $A(\widetilde{x}) = 0$. So, for any $\tilde{x} \in co(st\text{-}Exp(B_{\mathbb{X}}))$ we get $A(\tilde{x}) = 0$. Let $x \in B_{\mathbb{X}}$, then there exists $\{x_n\} \subset co(st\text{-}Exp(B_{\mathbb{X}}))$ such that $x_n \to x$. As $A(x_n) = 0 \ \forall n \in \mathbb{N}$, we get $A(x) = 0$. This implies $A = 0$. Therefore, $\mathcal{W}$ is strongly anti-coproximinal in $\mathbb{L}(\mathbb{X},\mathbb{Y})$. $\square$

Theorem 6.22. *Let $\mathbb{X}, \mathbb{Y}$ be a Banach spaces such that $B_{\mathbb{X}}$ is the closed convex hull of its strongly exposed points. Suppose that $\mathcal{W} \subset \mathbb{L}(\mathbb{X}, \mathbb{Y})$ is a closed subspace containing $\mathcal{F}(\mathbb{X}, \mathbb{Z})$ such that $ASE(\mathbb{X}, \mathbb{Z}) \cap \mathcal{W}$ is dense in $\mathcal{W}$. Then $\mathcal{W}$ is strongly anti-coproximinal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ if and only if $\mathbb{Z}$ is strongly anti-coproximinal in $\mathbb{Y}$.*

Proof. We only need to prove the necessary part. Suppose on the contrary that $\mathbb{Z}$ is not strongly anti-coproximinal in $\mathbb{Y}$. This implies there exists $y_0 \in \mathbb{Y}$ and $\epsilon \in [0, 1)$ such that $\mathbb{Z} \perp_B^{\epsilon} y_0$. Let $f \in S_{\mathbb{X}^*}$. Define $A : \mathbb{X} \to \mathbb{Y}$ such that $A(x) = f(x)y_0$. Let $T \in ASE(\mathbb{X}, \mathbb{Z}) \cap S_{\mathcal{W}}$ and let $M_T = \{\mu x_0 : \mu \in \mathbb{T}\}$, for some $x_0 \in S_{\mathbb{X}}$. As $\mathbb{Z} \perp_B^{\epsilon} y_0$, we get $Tx_0 \perp_B^{\epsilon} y_0 \implies Tx_0 \perp_B^{\epsilon} Ax_0$. Following Theorem 6.13, we get $T \perp_B^{\epsilon} A$. This implies $ASE(\mathbb{X}, \mathbb{Z}) \cap S_{\mathcal{W}} \perp_B^{\epsilon} A$. Since $ASE(\mathbb{X}, \mathbb{Z}) \cap \mathcal{W}$ is dense in $\mathcal{W}$, we get $\mathcal{W} \perp_B^{\epsilon} A$. This contradicts $\mathcal{W}$ is strongly anti-coproximinal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. □

In [21], several conditions are provided under which the set $ASE(\mathbb{X}, \mathbb{Y})$ is dense in a closed subspace of $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. One important condition is that $SE(B_{\mathbb{X}})$ is dense in $S_{\mathbb{X}^*}$ and $\mathbb{Y}$ has *quasi-β* property. The authors are referred to [1, 21] for the definition and detailed study about *quasi-β* property.

Lemma 6.5. *[21, Th. 3.1] Let $\mathbb{X}, \mathbb{Y}$ be Banach spaces. Suppose that $SE(B_{\mathbb{X}})$ is dense in $S_{\mathbb{X}^*}$ and that $\mathbb{Y}$ has quasi-β property. Then, for every closed subspace $\mathcal{W}$ of $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ containing all rank one operators, $ASE(\mathbb{X}, \mathbb{Y}) \cap \mathcal{W}$ is dense in $\mathcal{W}$.*

Applying Theorem 6.22 along with Lemma 6.5, the following result is immediate.

Corollary 6.6. *Let $\mathbb{X}, \mathbb{Y}$ be Banach spaces such that $SE(B_{\mathbb{X}})$ is dense in $S_{\mathbb{X}^*}$ and $B_{\mathbb{X}}$ is the closed convex hull of its strongly exposed points. Suppose that $\mathbb{Z}$ is a subspace of $\mathbb{Y}$ such that $\mathbb{Z}$ has quasi-β property. Then any closed subspace $\mathcal{W} \subset \mathbb{L}(\mathbb{X}, \mathbb{Z})$ containing $\mathcal{F}(\mathbb{X}, \mathbb{Z})$ is strongly anti-coproximinal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ if and only if $\mathbb{Z}$ is strongly anti-coproximinal in $\mathbb{Y}$.*

As c_0 is strongly anti-coproximinal in ℓ_∞ [55, Cor. 3.18], applying Theorem 6.22 we obtain the following result.

Corollary 6.7. *Let $\mathbb{X}$ be a Banach space such that $ASE(\mathbb{X}, c_0)$ is dense in $\mathbb{L}(\mathbb{X}, c_0)$ and $B_{\mathbb{X}}$ is the closed convex hull of its strongly exposed points. Then $\mathbb{L}(\mathbb{X}, c_0)$ is strongly anti-coproximinal in $\mathbb{L}(\mathbb{X}, \ell_\infty)$. In particular, $\mathbb{L}(\ell_p, c_0)$ is strongly anti-coproximinal in $\mathbb{L}(\ell_p, \ell_\infty)$, where $1 < p < \infty$.*

Whenever $\mathbb{X}$ satisfies Radon-Nikodým Property, it follows from [3, Th. 5] that for any given Banach space $\mathbb{Y}$, $ASE(\mathbb{X}, \mathbb{Y})$ is dense in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ and hence we obtain the following corollary of Theorem 6.22.

Corollary 6.8. *Let $\mathbb{X}$ has Radon-Nikodým Property and $\mathbb{Y}$ be any Banach space. Then $\mathbb{L}(\mathbb{X}, \mathbb{Z})$ is (strongly) anti-coproximinal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ if and only if $\mathbb{Z}$ is (strongly) anti-coproximinal in $\mathbb{Y}$.*

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Shamim Sohel
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