BOUNDARY ELEMENT ANALYSIS OF 2-D ELASTOSTATIC PROBLEMS

A Thesis

submitted for the partial fulfilment of the Degree of

MASTER OF ENGINEERING IN CIVIL ENGINEERING

Specialization In

Structural Engineering

By

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2022-2023

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CERTIFICATE OF RECOMMENDATION

This is to certify that the thesis entitled **“BOUNDARY ELEMENT ANALYSIS OF 2-D ELASTOSTATIC PROBLEMS”** submitted by **Sovan Mondal**, class roll: 002110402026, exam roll: M4CIV23012 and registration number: 135872 of 2016-17, in partial fulfilment of the requirements for the award of Master of Engineering Degree in Civil Engineering with specialization in “Structural Engineering” at Jadavpur University, Kolkata is an authentic work carried out by him under my supervision and guidance.

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I, Sovan Mondal, Master of Engineering (Structural Engineering), Jadavpur University, Faculty of Engineering and Technology, hereby declare that the work being presented in the thesis work entitled, **"BOUNDARY ELEMENT ANALYSIS OF 2-D ELASTOSTATIC PROBLEM"**, is authentic record of work and has been carried out at the Department of Civil Engineering, Jadavpur University, Kolkata under Dr. Arup Guha Niyogi, Professor of Department of Civil Engineering, Jadavpur University. The work contained in the thesis has not yet been submitted in part or full to any other university or institution or professional body for award of any degree or diploma or any fellowship.

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ACKNOWLEDGEMENT

I would like to express my gratitude to my parents who encouraged my mathematical and scientific interest from an early age.

I express my deep sense of respect and gratitude to Dr. Arup Guha Niyogi, Professor of Department of Civil Engineering, Jadavpur University under whose supervision the work has been done. He has taken the strain of giving valuable suggestions to me from time to time. It is his never-ending enthusiasm and constant urge for excellence that has given me to bring this thesis work to completion.

I extend my thanks to all the faculty members of Department of Civil Engineering, Jadavpur University for providing their moral support, encouragement, and the necessary infrastructure for the work. I also acknowledgement the cooperation and assistance provided by my friends during this period.

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Reg. No.: 135872 of 2016-17

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List of Symbols

Symbol	Description
X_1, X_2, X_3	Global coordinate system
σ_{ij}	Stress components
ε_{ij}	Strain components
E	Young modulus
G	Shear modulus
ν	Poisson's ratio
λ	Lame's constant
u_k	Displacement components
u_k^*	Displacement components of fundamental solutions
p_k	Surface traction components
p_k^*	Surface traction components of fundamental solutions
b_k	Body forces components
Δ^i	Dirac delta function
Ω	Integration domain
Γ	Boundary domain
H, G and B	Boundary elements matrices
N	No of elements

Abstract

Practical engineering problems are challenging to solve analytically due to their complex domains and intricate boundary conditions. Thus, to the desired level of accuracy, such engineering problems are frequently solved numerically. There are other approximation methods for making numerical solutions, however in my thesis work, I used the boundary element method (BEM), where the problem's dimensionality is reduced by one. Two-dimensional elements can be used to model three-dimensional objects, and line elements can be used to model two-dimensional domains. In comparison to domain-specific approaches like finite element or finite difference methods, BEM takes less processing time and storage space. Hence, boundary element method has been preferred here over other domain methods. By using the boundary element method, an attempt has been made to solve two-dimensional plane stress and plain strain elasticity problems using constant elements.

CHAPTER 1

INTRODUCTION

The Boundary Element Method (BEM) constitutes a technique for analyzing the behavior of mechanical systems and especially of engineering structures subjected to external loading. The term loading is used here in the general sense, referring to the external source which produces a non-zero field function that describes the response of the system (temperature field, displacement field, stress field, etc.), and it may be heat, surface tractions, body forces, or even non-homogeneous boundary conditions, e.g. support settlement.

Study of the behavior of complex structures are conducted today mostly using computer methods. The reason is quite obvious, the low cost of the numerical versus the expensive experimental simulation. Numerical modeling can be used to study a wide variety of loadings and geometries of a structure and to determine the optimum design solution, before proceeding to its construction.

The method adopted for the numerical analysis of structures during the last 50 years is mainly the Finite Element Method (FEM). It is the method with which realistic problems of engineering are being solved, that is the analysis of structural elements of arbitrary geometry, arbitrary loading, variety of constitutive relations, with linear or non-linear behavior, in two or three dimensions. Justifiably, the FEM has been valued during the last 30 years as a modern computational tool.

A reasonable question to ask is "why do we need the BEM since we already have the FEM that solves engineering problems?". The answer is that a modeling with finite elements can be ineffective and laborious for certain classes of problems. So, the FEM, despite the generality of its application in engineering problems, is not free of drawbacks. The most important of which are:

(i) Discretization is over the entire domain occupied by the body. Hence, generation and inspection of the finite element mesh exhibit difficulty and are both laborious and time consuming, especially when the geometry of the body is not simple. For example, when there are holes, notches or corners, mesh refinement and high element density is required at these critical regions of large solution gradients (Fig. 1.1a).

(ii) Modification of the discretized model to improve the accuracy of the solution or to reflect design changes can be difficult and requires a lot of effort and time.

(iii) For infinite domains, e.g. half-space or the complementary domain to a finite one, fabrication of fictitious closed boundaries is required in order to apply the FEM. This reduces the accuracy and sometimes may result in spurious or incorrect solutions.

(iv) For problems described by differential equations of fourth or higher order (i.e., plate equations, or shell equations of sixth, eighth or higher order), the conformity requirements demand such a tedious job that FEM may become impractical.

(v) Although the FEM computes accurately the field function, which is the unknown of the problem, it is ineffective in determining its derivatives. The accuracy drops considerably in areas of large gradients.

Drawbacks (i) and (ii) can be overcome by using updated releases of advanced professional finite element software, such as NASTRAN, which are equipped with automatic and adaptable mesh generators. Essentially, the task of generating a finite element mesh is a difficult geometric problem, and in several cases, it may prove to be far more difficult than the physical problem which is to be solved by the FEM. The new disadvantage, though, is that the codes for creating FEM meshes are closed, and any effort to produce them requires specialized knowledge from a different scientific area. An attempt, by the engineer, to learn the necessary material will only distract him from his original goal, which is to solve a physical problem.

On the contrary, the boundary element method possesses many advantages, the most important of which are:

(i) Discretization is only over the boundary of the body, making the numerical modeling with the BEM easy (see Fig. 1.1b) and reducing the number of unknowns by one order. Thus, a remodeling to reflect design changes becomes simple.

(ii) For infinite domains, the problem is formulated simply as an exterior one. Apparently, the fundamental solution has to satisfy some conditions at infinity, such as Sommerfeld's radiation condition for problems in dynamics. In this manner, computer programs developed for finite domains can be used, with just few modifications, to solve problems in infinite domains. This is not possible with the FEM.

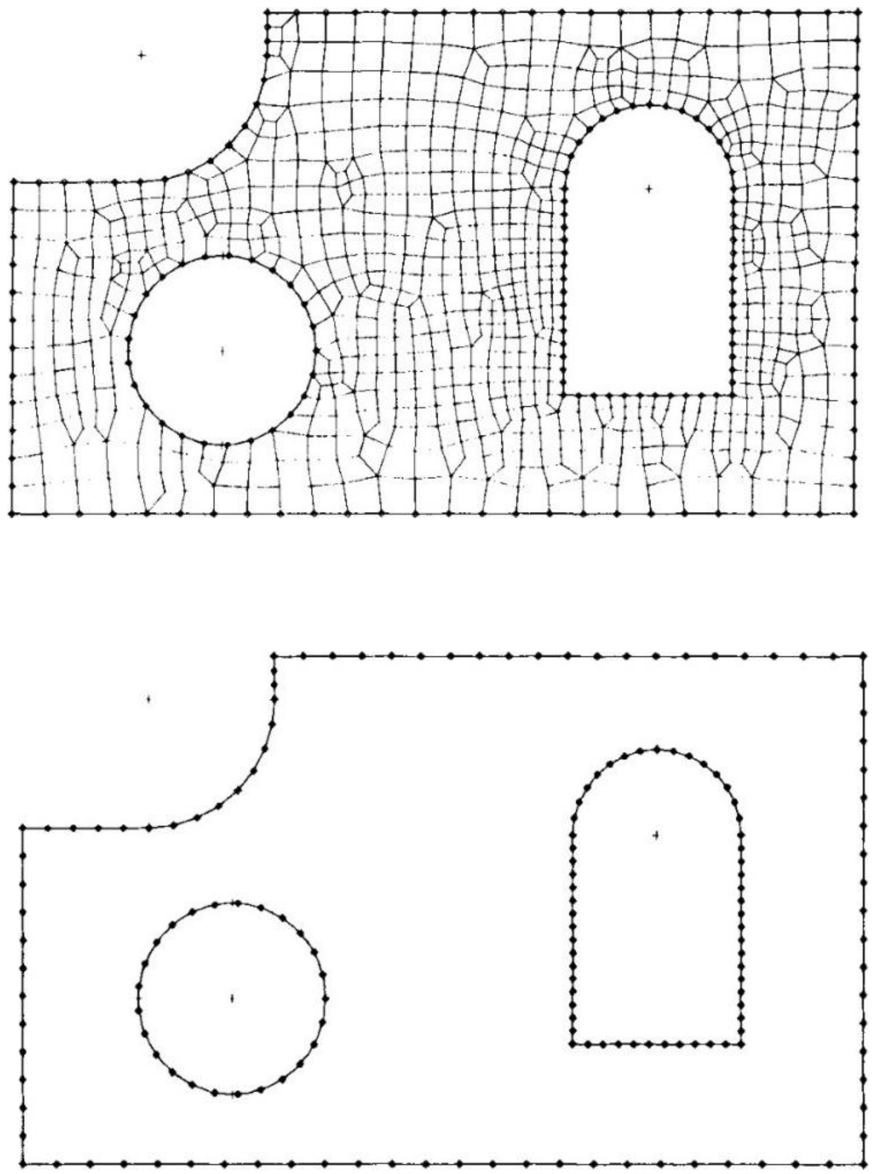


Fig 1.1 (a) Domain (FEM) and (b) boundary (BEM) discretization.

(iii) The method is particularly effective in computing the derivatives of the field function (e.g., fluxes, strains, stresses, and moments). It can easily handle concentrated forces and moments, either inside the domain or on the boundary.

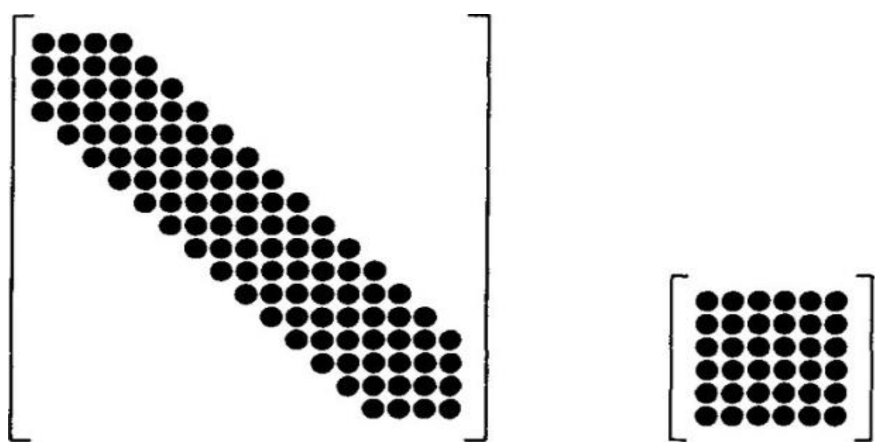
(iv) The BEM allows evaluation of the solution and its derivatives at any point of the domain of the problem and at any instant in time. This is feasible because the method uses an integral representation of the solution as a continuous mathematical expression, which can be differentiated and utilized as a mathematical formula. This is impossible with the FEM, since the solution is obtained only at the nodal points.

(v) The method is well suited for solving problems in domains with geometric peculiarities, such as cracks.

At its current stage of development, the BEM exhibits the following main disadvantages:

(i) Application of the BEM requires the so-called fundamental solution. The method cannot be used for problems whose fundamental solution is either not known or cannot be determined. Such are, for example, problems described by differential equations with variable coefficients. The method is obviously not applicable to non-linear problems for which the principle of superposition does not hold. In this case, a BEM model produces domain integrals that can be computed by discretizing the domain, but this, of course, spoils the pure boundary character of the method. Intense research has been conducted in an effort to overcome the aforementioned disadvantages.

(ii) The numerical implementation of the BEM results in systems of linear algebraic equations whose coefficient matrices are fully populated and non-symmetric. In a FEM model, however, the corresponding matrices have some very nice properties, they are banded and symmetric. This drawback of the BEM is counterbalanced by the much smaller dimensions of its matrices. The general format of the coefficient matrices for a FEM and BEM model is shown graphically in Fig. 1.2.



FEM BEM

Fig. 1.2 Coefficient matrices for FEM and BEM.

CHAPTER 2
LITERATURE REVIEW

About BEM

In early 1980's the BEM was known as boundary integral equation method (BIEM). The base work of this work was done by **Green [1]** in 1828, when he used Green's function to solve Laplace type equations, he established the integral representation of the solution for the Dirichlet and Neumann problems.

In 1872 **Betti [2]** used a general method to integrate the elasticity equations and derivation of integral form solution. It is considered to be a straight forward development of Green's theorem in Navier elasticity equations.

Betti's reciprocal theorem was utilized by **Somigliana [3]** in 1885 to derive an integral representation of solution for elasticity problem.

Singular boundary integral equations were initially used by **Fredholm [4]** to determine unknown boundary quantities and solve potential problems at the turn of the twentieth century. The boundary conditions for a well-posed mathematical physics problem were determined using this method as a mathematical tool but not as a solution method. This is logical because the analytical solution of the generated singular integral equations is not always attainable. The foundation of indirect BEM was laid by **Fredholm** integral equations.

Direct BEM is used when the unknown boundary quantities have a direct physical or geometrical meaning. Other BEM formulations in which the unknown boundary quantities do not have geometrical or direct physical meaning, such formulations fall into the class of indirect BEM.

Jawson [5] and **Symm [6]** discretized the integral equations of a two-dimensional potential problem represented as Laplace equation into straight line elements with constant potential functions. Except for some singular integrals that were integrated analytically, the elements are described in terms of nodal points, with integrations made using Simpson's rule. As the functions employed to formulate the problem are not imaginary and may be differentiated or integrated to determine physical values, their approach is referred to as **"semi-direct"**, proposed a more generic formulation based on Green's third identity, with potentials and derivatives as the boundary unknown, and the results of this formulation were expressed **[5]** and **[7]**.

For solving of the Neumann-type boundary value problem, **Hess and Smith [8]** described the technique for developing a completely new method of solving general engineering problems. However, using the **indirect boundary element** technique, Hess and Smith constructed a number of strong programmes to obtain the Laplace type boundary value problem solution and applied it to potential flow and arbitrary bodies. To increase computer performance, the influence coefficients were analytically calculated in case of two- and three-dimensional instances, the equations were solved by iteration using the Gauss-Seidel method, and multipole expansions were used to determine elemental effect placed far away from the real node.

It's impossible to say who was the first to adopt the "direct" way of analysis. From an engineering standpoint, **Cruse and Rizzo's [9]** work in elastostatics is thought to have inspired the approach. For engineering and physical science problems, the direct method is the most appealing. Rizzo's work was fundamental since he was the first to see the strong resemblance between potential theory and classical elasticity theory and to devise a numerical method for solving the problem. He discretized the boundary using straight-line elements, with functions (displacements and tractions in this case) considered constant over each element. Except for singular integrals, Simpson's rule was applied.

In 1978 the BEM is observed connecting other numerical techniques example given finite differences and finite elements, notably through the work of Brebbia's work and collaborations**.**

The boundary integral equations (BIE) formulation methods can be divided into three categories as follows:

The indirect BEM: This method is based on the use of fictional density functions to formulate the problem. Despite the fact that these density functions have no physical relevance, we can derive the actual displacements and stresses by integrating them. As a result, this method is referred to as "indirect" because actual displacements and stresses are not used right once.

Semi-direct BEM: In this method, the functions used to formulate the problem are linked to stress functions, which have a little more physical significance than fictitious density functions. To compute displacements and stresses, these functions can be separated or integrated.

Direct BEM: In this method, real physical quantities such as displacements and tractions are employed to formulate the integral equations from the start. Using reciprocal work theorems, partial differential equations of elasticity derived from equilibrium equations, constitutive relationships, and relationship of strain and displacement are converted into integral equations that can be applied to the boundary. This strategy is simpler to comprehend because it involves real quantities rather than a hypothetical problem. As a result, this strategy will be employed.

The BEM's efficiency drew researchers in and encouraged them to continue developing the technology. A large number of literature papers on BEM for tackling a wide variety of engineering challenges were found in the late 1980s. The fundamental goal of new improvements in BEM is to overcome any method flaws that may arise. They attempt to tackle complex time-dependent problems, as well as linear and nonlinear problems for which the underlying solution is unknown. The resulting integral solution for all of these types of problems involves domain integrals, which makes the method more difficult to apply.

The dual reciprocity method (DRM) and the analogue equation method (AEM) are the most efficient techniques for successfully overcoming most of the obstacles such as domain integration while maintaining the BEM's purely boundary only feature.

About 2-D Elastostatics

The boundary element method, in fact, had evolved from the integral formulation of the theory of elasticity. **Fredholm [4]**, at the turn of the twentieth century, presented systematic and detailed integral equation formulation for elasticity problems. Further important progresses were made by **Muskhelishvili [10]**, **Mikhlin [11]**, **Smirnov [12]**, **Kupradze [13]**, **Kellogg [14]** and **Massonnet [15]**. The advent of high-speed digital computers provided a stimulus to a renewed interest in the integral equation formulation and their computer oriented numerical solution. **Rizzo [16]**, for the first time, successfully converted the system of singular integral equations of elasticity to a system of algebraic equations whose solution represented an approximate solution of the elastostatic problem. The present-day boundary element method had evolved from Rizzo's formulation.

The application of boundary elements to study linear elastostatics problems was based on the direct approach and followed the notation initiated by Alarcon, Brebbia and Dominguez in references **[17]** and **[18]**. The direct formulation of boundary elements for elastostatics was presented by **Rizzo** in 1967 **[16]** following the work of Jaswon. The basic integral representation known as Somigliana's identity [19] was taken to the boundary which was then discretized into constant elements in a way similar to that previously used for potential problems. **Cruse and Rizzo [9]** and **Cruse [20]** extended the formulation to elastodynamics.

In linear elasticity, the equilibrium equation in terms of displacement, in absence of body forces, was that of Navier equation, whose analytical solution could be obtained by means of potential formulations which simplify the elastic problem by yielding uncoupled governing equations in terms of some potential unknowns. These alternative formulations had allowed to solve a great variety of interesting problems such as **Boussinesq** and **Kelvin [21]**, among others. As regards numerical solutions, classical methods, such as finite-difference, derived solutions from practical problems based, for example, on the potential formulations of the Airy's stress function – for 2D elasticity problems – or the Prandtl's function – for torsion in prismatic bars. **Wang et. al. [22]** resume the set of potential formulations related to linear elasticity field and their applications.

In the year of 2022 **Ney Augusto Dumont [23]**, in a paper introduced the collocation boundary element method for three-dimensional static elasticity problems. This method employed Gauss-Legendre quadratures for high computational accuracy, regardless of geometry or topology. It handled quasi-singularities efficiently with a limited number of quadrature points, and addresses various mathematical aspects, including improper and singular integrals. Numerical results and convergence analysis were presented in a second paper **[25]**.

Objective

The objective of the present work is to learn the principles of Boundary Element Method (BEM) and implement it in the field of elasticity, with the aim of understanding how BEM can enhance our ability to analyze and solve complex elasticity problems efficiently and accurately. This study will involve theoretical learning of BEM, its numerical implementation, and its real-world application in elasticity-related scenarios, leading to a comprehensive understanding of BEM's potential in this domain.

Scope of work

This work is restricted to

1. Use of Boundary Element Method in 2D elasticity problems using the simplest, i.e., constant boundary elements to explore the methodology and implementation clearly.

2. Solution of plane stress and plane strain problems, using a common program.

Apart from computing the unknown boundary stresses and displacements, the program is capable of computing the stresses and displacement at internal points analytically, once the external parameters are all known.

CHAPTER 3
THEORY

3.1 Two Dimensional Elastostatics

This chapter deals with the applications of boundary elements to solve two dimensional elastostatics problems. The basic equations of elasticity are reviewed, first pointing out that boundary elements for these cases are based on the plane strain approach but can be extended to plane stress if the elastic coefficients are replaced by the corresponding equivalent values, as will be seen below.

The boundary element formulation of elastostatics is substantially different from the one for potential problems as its unknowns are in vector rather than scalar form. This means that at a particular point there are two (or three in three dimensions) components of displacements or tractions. The associated fundamental solution also represents a point load acting in a given direction. All this implies that the boundary element formulation for elastostatics is more complex than potential problems although it involves the same basic steps.

It seems appropriate to start by developing the necessary theory and then the computer code corresponding to the simplest case, i.e., the one with constant elements. Hence while constant or linear elements can be satisfactorily applied for many elasticity problems, this work was restricted to constant elements only for reducing the challenge which might result in not so accurate estimate of stress might result.

3.2 Plane Stress and Plane Strain Problems

Two dimensional problems are divided into two types, plate stretching or, rather commonly called, plane stress problems and plane strain problems, depending on how the solid is restrained in the direction perpendicular to the plane under study.

To understand the difference between these two states, we consider the prismatic homogeneous solid shown in Fig. 3.1. The end surfaces are defined by the planes at $x_3 = \pm\left(h/2\right)$ and the cylindrical surface by $x_1 = x_1(\Gamma)$, $x_2 = x_2(\Gamma)$, where Γ is the arc length along the boundary curve.

The basic assumptions for plane stress are:

i. that the body is thin, i.e., h is small by comparison with the representative dimensions along x_1 or x_2,

ii. there are no tractions acting at the end surfaces, i.e., at $x_3 = \pm\left(h/_2\right)$, $p_j = 0$,

iii. the body forces are acting on $x_1 - x_2$ planes and independent of x_3 i.e. $b_3 = 0$ and b_1 b_2 are functions of x_1 and x_2 only,

iv. the forces acting on the cylindrical body are planar and independent of x_3, i.e. $p_3 = 0$ and p_1 p_2 are functions of x_1 and x_2.

Under these assumptions it is assumed that σ_{33}, σ_{31} and σ_{32} are all small in comparison with σ_{11} σ_{22} σ_{12} and that the variation of the latter with respect to x_3 is negligible. Hence one assumes:

$$\sigma_{33} = \sigma_{31} = \sigma_{32} = 0 \qquad 3.1$$

and σ_{11}, σ_{22} and σ_{12} are function of x_1, x_2 only.

$$\sigma_{11}(x_1, x_2);\ \sigma_{22}(x_1, x_2);\ \sigma_{12}(x_1, x_2);$$

It should be noticed however that although these assumptions are reasonable in engineering practice, they are only approximate as they violate the compatibility equations.

Equation 3.1 is written in function of displacements and one can have an alternative statement, i.e. the displacement components are functions of x_1 x_2 only,

i.e., $u_1 = u_1\ (x_1, x_2);\ \ u_2 = u_2\ (x_1, x_2)$ 3.2

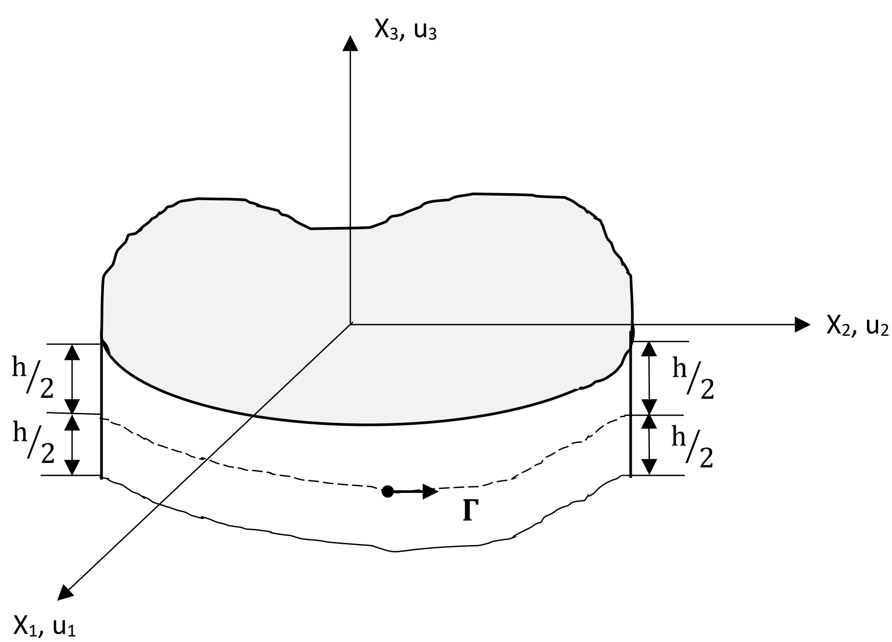


Figure 3.1 Prismatic solid

Notice that $u_3 \neq 0$ and the ε_{33} strains can be determined in function of the $\sigma_{11}, \sigma_{22}, \sigma_{12}$ stresses.

Plane Strain Condition: This case usually represents the behavior of long structures such as tunnels and for this the displacements in the normal direction can be assumed to be zero (i.e. on the end faces).

The plane strain assumptions are,

i. The end faces displacements u_3 are zero as they are restrained to move normally because the thickness is large in comparison with the representative dimensions in x_1 x_2 directions.
ii. The body and surface forces acting on the cylindrical surface have no x_3 component and are independent of x_3.

In this case in addition to $u_3 = 0$, one can assume that the in-plane displacements u_1 and u_2 are independent of x_3. Hence

$$u_1 = u_1(x_1, x_2); \quad u_2 = u_2(x_1, x_2); \quad u_3 = 0 \qquad 3.3$$

This means that some of the resulting strains will also be zero, i.e.

$$\varepsilon_{33} = \varepsilon_{31} = \varepsilon_{32} = 0 \qquad 3.4$$

and the others independent of x_3, i.e. $\varepsilon_{11}(x_1, x_2)$, $\varepsilon_{22}(x_1, x_2)$, $\varepsilon_{12}(x_1, x_2)$.

For this case $\sigma_{33} \neq 0$ and can be determined from the value of the other components.

3.3 Constitutive Relations

One can now expand the three-dimensional stress-strain relationships for an isotropic body in terms of the non-zero components for plane strain or plane stress, i.e.

$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33})$$

$$\varepsilon_{22} = \frac{1}{E}(-\nu\sigma_{11} + \sigma_{22} - \nu\sigma_{33})$$

$$\varepsilon_{33} = \frac{1}{E}(-\nu\sigma_{11} - \nu\sigma_{22} + \sigma_{33})$$

$$\varepsilon_{12} = \frac{1}{2G}\sigma_{12} \qquad 3.5$$

The relationships for plate stretching can be obtained by setting $\sigma_{33} = 0$ in 3.5 and considering only the planar components ε_{11}, ε_{22} and ε_{12}, i.e.

$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22})$$

$$\varepsilon_{22} = \frac{1}{E}(-\nu\sigma_{11} + \sigma_{22})$$

$$\varepsilon_{12} = \frac{1}{2G}\sigma_{12} \qquad 3.6$$

The value of ε_{33} can be obtained a posteriori from the third relationship in (3.5), i.e.

$$\varepsilon_{33} = \frac{1}{E}(-\nu\sigma_{11} - \nu\sigma_{22}) \qquad 3.7$$

Equations (3.6) can be inverted to produce,

$$\sigma_{11} = \frac{E}{(1-\nu^2)}(\varepsilon_{11} + \nu\varepsilon_{22})$$

$$\sigma_{22} = \frac{E}{(1-\nu^2)}(\varepsilon_{22} + \nu\varepsilon_{11})$$

$$\sigma_{12} = \frac{E}{(1+\nu)}\varepsilon_{12} \qquad 3.8$$

The plane strain equations can be found by first eliminating ε_{33} from Eq. (3.5), i.e.

$$\varepsilon_{33} = \frac{1}{E}(-\nu\sigma_{11} - \nu\sigma_{22} + \sigma_{33}) \equiv 0 \qquad 3.9$$

which gives

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) \qquad 3.10$$

Substituting for σ_{33} in Eq. (3.5) one obtains the following expressions,

$$\varepsilon_{11} = \frac{1}{E}[(1-\nu^2)\sigma_{11} - \nu(1+\nu)\sigma_{22}]$$

$$\varepsilon_{22} = \frac{1}{E}[-\nu(1+\nu)\sigma_{11} + (1-\nu^2)\sigma_{22}]$$

$$\varepsilon_{12} = \frac{1}{2G}\sigma_{12} \qquad 3.11$$

The inverse of these expressions is

$$\sigma_{11} = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_{11} + \nu\varepsilon_{22})$$

$$\sigma_{22} = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_{11} + (1-\nu)\varepsilon_{22}]$$

$$\sigma_{12} = 2G\,\varepsilon_{12} \qquad 3.12$$

The value of ε_{33} if required can be obtained from (3.10).

Notice that it is sometimes more convenient to express the first two relationships above using Lamé's constant $\lambda = {}^{\nu E}/_{[(1+\nu)(1-2\nu)]}$, so that

$$\sigma_{11} = \frac{\lambda}{\nu}\ [(1-\nu)\varepsilon_{11} + \ \nu\varepsilon_{22})$$

$$\sigma_{22} = \frac{\lambda}{\nu}\ [\ \nu\varepsilon_{11} + (1-\nu)\varepsilon_{22}] \qquad 3.13$$

One can pass from equations (3.8) for plane stress to equations (3.12) for plane strain simply by replacing E and ν in the first equation by two equivalent values, E′ and ν′, such that

$$E' = \frac{E}{(1-\nu^2)}; \qquad \nu' = \frac{\nu}{1-\nu} \qquad 3.14$$

where the value of G remains the same.

This interesting relationship means that one can implement a plate stretching program and by transforming the elastic constant data in accordance to (3.14) solve also a plane strain problem. This is done in finite element analysis. Conversely in boundary elements one works with plane strain problems as the fundamental solution is known for this type of problem and then plate stretching problems can be solved using the inverse relationship to (3.14), i.e.

$$E' = (1-\nu^2)E; \ \ \nu' = \frac{\nu}{1+\nu} \qquad 3.15$$

3.4 Boundary Integral Formulation

The governing integral equations for elastostatics will be deduced using considerations of weighted residual. The fact that some terms which are assumed to be approximate may not be so does not detract from the use of these concepts as a general way of producing the required statement. The concepts are very similar to those used in virtual work.

Consider first that one desires to minimize the errors involved in the numerical approximation of the governing equations of elastostatics, i.e.

$$\sigma_{kj,j} + b_k = 0 \quad \text{in } \Omega \qquad 3.16$$

which usually have to satisfy the following conditions

(i) Essential or displacement conditions

$$u_k = \bar{u}_k \qquad \text{on } \Gamma_1 \qquad 3.17$$

(ii) Natural or traction conditions

$$p_k = \sigma_{kj} n_j = \bar{p}_k \qquad \text{in } \Gamma_2 \tag{3.18}$$

Consider first that we are only interested in minimizing (3.16). To this end one can weight each of these equations by displacement type functions u_k^* and orthogonalize the product, i.e.

$$\int_\Omega \left(\sigma_{kj,j} + b_k\right) u_k^* \, d\Omega = 0 \tag{3.19}$$

If we carry out the integration by parts on the first term of this equation and group the corresponding terms together, one finds the following expression

$$-\int_\Omega \sigma_{kj} u_k^* \, d\Omega + \int_\Omega b_k u_k^* \, d\Omega = -\int_\Gamma p_k u_k^* \, d\Gamma \tag{3.20}$$

Integrating by parts one finds the adjoint of the equation (3.16), i.e.

$$\int_\Omega \sigma_{kj,j}^* u_k \, d\Omega + \int_\Omega b_k u_k^* \, d\Omega = -\int_\Gamma p_k u_k^* \, d\Gamma + \int_\Gamma u_k p_k^* \, d\Gamma \tag{3.21}$$

This expression corresponds to Betti's reciprocal theorem (notice that $\sigma_{kj,j}^* = -b_k^*$) which is sometimes used as the starting point for the boundary integral formulation.

It may be noted that the two terms on the right-hand side are integrals on the Γ surface of the body. Let us now consider that the boundary is divided into two parts Γ_1 and Γ_2, and on each of them the boundary conditions (3.17) and (3.18) apply. Hence on can write (3.21)

$$\int_\Omega \sigma_{kj,j}^* u_k d\Omega + \int_\Omega b_k u_k^* \, d\Omega = -\int_{\Gamma_1} p_k u_k^* \, d\Gamma - \int_{\Gamma_2} \bar{p}_k u_k^* \, d\Gamma + \int_{\Gamma_1} \bar{u}_k p_k^* \, d\Gamma + \int_{\Gamma_2} u_k p_k^* \, d\Gamma \tag{3.22}$$

The bars represent known values of displacements u_k and tractions p_k components. One can now integrate again by parts trying to return to equation (3.19) but we will find that the resulting expression is slightly different as we have now imposed boundary conditions in Γ_1 and Γ_2. Integrating by parts twice the first integral in (3.22) one obtains,

$$\int_\Omega \left(\sigma_{kj,j} + b_k\right) u_k^* \, d\Omega = \int_{\Gamma_2} (p_k - \bar{p}_k) u_k^* \, d\Gamma + \int_{\Gamma_1} (\bar{u}_k - u_k) p_k^* \, d\Gamma \tag{3.23}$$

This expression is a generalized statement that can be used to obtain the boundary integral equations. Having established this starting principle one can now return to Eq. (3.22) and use as weighting functions the fundamental solution was obtained for a point load $\mathrm{b_l} = \Delta^{\mathrm{i}}$ along the direction of the unit vector $\mathrm{e_l}$,

i.e.

$$\sigma^*_{\mathrm{ij,j}} + \Delta^{\mathrm{i}}\mathrm{e_l} = 0 \qquad 3.24$$

The fundamental solution may be written as before, i.e.,

$$\mathrm{u}^*_{\mathrm{k}} = \mathrm{u}^*_{\mathrm{lk}}\mathrm{e_l}$$

$$\mathrm{p}^*_{\mathrm{k}} = \mathrm{p}^*_{\mathrm{lk}}\mathrm{e_l} \qquad 3.25$$

where $\mathrm{u}^*_{\mathrm{lk}}$, $\mathrm{p}^*_{\mathrm{lk}}$ are k components of displacements and tractions due to a unit point load in the *l* direction. The first integral in Eq. (3.22) for a particular direction $\mathrm{e_l}$ of the unit load becomes

$$\int_{\Omega} \sigma^*_{\mathrm{kj,j}}\, \mathrm{u_k}\, \mathrm{d}\Omega = \int_{\Omega} \sigma^*_{\mathrm{ij,j}}\, \mathrm{u_l}\, \mathrm{d}\Omega$$

$$= -\int_{\Omega} \Delta^i \mathrm{u_l} e_l\, \mathrm{d}\Omega = -u_l^i e_l \qquad 3.26$$

where u_l^i represents the *l* component of the displacement at the point *i* of application of the load.

Equation (3.22) can now be written to represent the three separate components of the displacement at *i* by taking the three directions of the point load at '*i*'.

Independently, i.e.,

$$u_l^i + \int_{\Gamma_1} \bar{\mathrm{u}}_{\mathrm{k}}\mathrm{p}^*_{\mathrm{lk}}\, \mathrm{d}\Gamma + \int_{\Gamma_2} \mathrm{u_k}\mathrm{p}^*_{\mathrm{lk}}\, \mathrm{d}\Gamma$$

$$= \int_{\Gamma_1} \mathrm{p_k}\mathrm{u}^*_{\mathrm{lk}}\, \mathrm{d}\Gamma + \int_{\Gamma_2} \bar{\mathrm{p}}_{\mathrm{k}}\mathrm{u}^*_{\mathrm{lk}}\, \mathrm{d}\Gamma + \int_{\Omega} \mathrm{b_k}\mathrm{u}^*_{\mathrm{lk}}\, \mathrm{d}\Omega \qquad 3.27$$

Notice that when one applies a unit point load along a particular direction "l", the tractions and displacements at any point in the domain have components along the three or two directions (Eq. (3.25)) while terms of the type $\sigma_{\mathrm{lj,j}}$ only are different from zero along the direction of the load.

This leads to the fact that for a given direction '*l*' at the point the first term of (3.22) only produces displacements along the direction of the load (first term of (3.27). The rest of the terms however include products for all the components.

Equation (3.27) can be written in a more compact way if one considers the two parts of the boundary together (i.e. $\Gamma = \Gamma_1 + \Gamma_2$) and applies the boundary conditions at a later stage. In this case (3.27) becomes

$$u_l^i + \int_\Gamma \mathrm{u_k p_{lk}^*}\, d\Gamma = \int_\Gamma \mathrm{p_k u_{lk}^*}\, d\Gamma + \int_\Omega \mathrm{b_k u_{lk}^*}\, d\Omega \qquad 3.28$$

This equation is known as Somigliana's identity and gives the value of the displacements at any internal points in terms of the boundary values $\mathrm{u_k}$ and $\mathrm{p_k}$, the forces throughout the domain and the known fundamental solution. Equation (3.28) is valid for any particular point ‘i’ where the forces are applied.

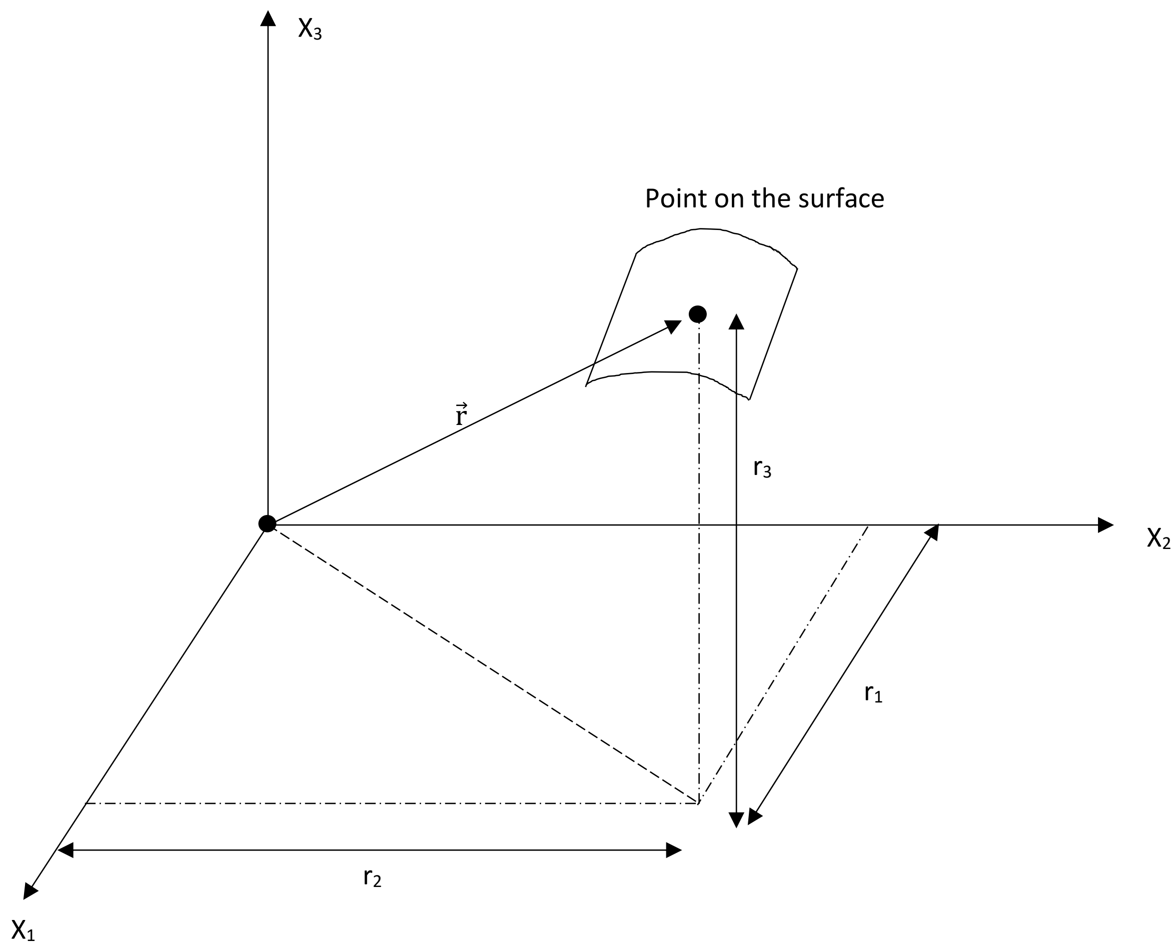


Figure 3.2 Interpretation of the components of the distance vector $\vec{r}$

3.4.1 Boundary Points

Somigliana's identity gives the displacement at any internal point once u_k and p_k are known at every boundary point and consequently only when the boundary value problem has been solved the values at the internal points can be calculated. However, since equation (3.28) is valid for every point in Ω including Γ. A boundary integral expression can be obtained by taking (3.28) to the boundary. This expression is applied at different points on the boundary to produce a system of equations which once solved give the boundary values.

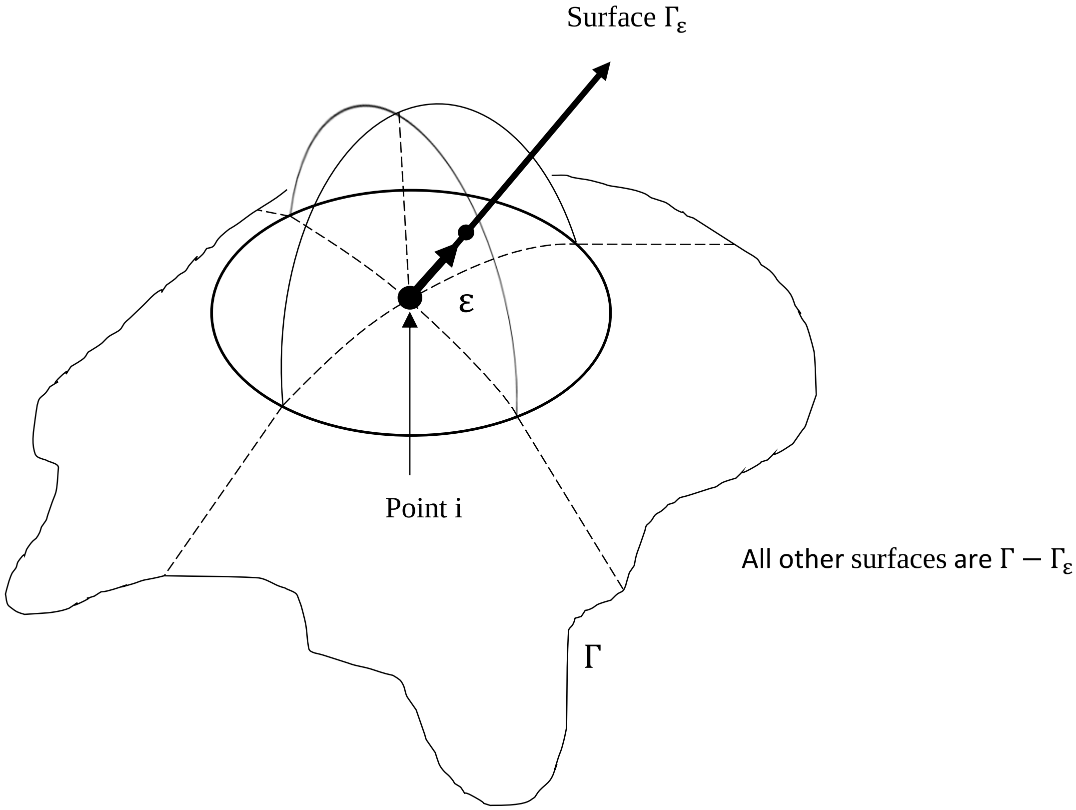


Figure 3.3 Full boundary surface Γ_ε assumed hemispherical for integration purposes

When 'i' is taken to the boundary, however, the integrals have a singularity and we need to analyze this behavior in the same way as it was done for potential problems. If we consider that the boundary is smooth at 'i' one can supplement it by a hemisphere (figure 3.3) with centre at 'i' and a small radius ε which will afterwards be taken to the limit, i.e. $\varepsilon \rightarrow 0$.

There are two types of boundary integrals in equation (3.28). Consider first the one on the right hand side and write it in function of Γ_ε surface of the hemisphere, i.e.,

$$\int_{\Gamma} \mathrm{p_k u^*_{lk}}\, \mathrm{d}\Gamma = \lim_{\varepsilon\to 0}\left\{\int_{\Gamma-\Gamma_\varepsilon} \mathrm{p_k u^*_{lk}}\, \mathrm{d}\Gamma\right\} + \lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon} \mathrm{p_k u^*_{lk}}\, \mathrm{d}\Gamma\right\} \qquad 3.29$$

The first integral on the right-hand side of (3.29) will simply become an integral on the whole boundary Γ when $\varepsilon \to 0$. The second integral can be written as,

$$p_k^l \lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon} \mathrm{u^*_{lk}}\, \mathrm{d}\Gamma\right\} \qquad 3.30$$

Noticing that the fundamental solution is of order $1/\varepsilon$ and the surface integral in (3.30) will produce a ε^2, one can conclude that (3.30) will tend to zero as $\varepsilon \to 0$, i.e.

$$\lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon} \mathrm{u^*_{lk}}\, \mathrm{d}\Gamma\right\} \equiv 0 \qquad 3.31$$

In other words the integral investigated is not affected by the singularity at 'i'.

The left hand side integral in (3.28) however behaves differently. If one writes it as

$$\int_{\Gamma} \mathrm{u_k p^*_{lk}}\, \mathrm{d}\Gamma = \lim_{\varepsilon\to 0}\left\{\int_{\Gamma-\Gamma_\varepsilon} \mathrm{u_k p^*_{lk}}\, \mathrm{d}\Gamma\right\} + \lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon} \mathrm{u_k p^*_{lk}}\, \mathrm{d}\Gamma\right\} \qquad 3.32$$

one can see that the limit of the last integral can be written as

$$\lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon} \mathrm{u_k p^*_{lk}}\, \mathrm{d}\Gamma\right\} = u_k^l \lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon} \mathrm{p^*_{lk}}\, \mathrm{d}\Gamma\right\} \qquad 3.33$$

The $\mathrm{p^*_{lk}}$ values are now of $1/\varepsilon^2$ order while the terms resulting from integration over the surface of the hemisphere are of order ε^2. Hence the integral (3.33) does not vanish when $\varepsilon \to 0$ but produces a free term. By substituting the values of $\mathrm{p^*_{lk}}$ and integrating over Γ_ε one finds

$$\lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon} \mathrm{p^*_{lk}}\, \mathrm{d}\Gamma\right\} = -\frac{1}{2}\,\delta_{lk} \qquad 3.34$$

Hence the left-hand side integral (3.32) can be written in the limit as

$$\int_{\Gamma} \mathrm{u_k p^*_{lk}}\, \mathrm{d}\Gamma - \frac{1}{2}\,\delta_{lk} u_k^i = \int_{\Gamma} \mathrm{u_k p^*_{lk}}\, \mathrm{d}\Gamma - \frac{1}{2}\, u_l^i \qquad 3.35$$

where the integral on Γ is defined in the sense of the Cauchy Principal value.

Therefore for boundary points equation (3.28) transforms into

$$c_{lk}^i u_k^i + \int_{\Gamma} \mathrm{u_k p^*_{lk}}\, \mathrm{d}\Gamma = \int_{\Gamma} \mathrm{p_k u^*_{lk}}\, \mathrm{d}\Gamma + \int_{\Omega} \mathrm{b_k u^*_{lk}}\, \mathrm{d}\Omega \qquad 3.36$$

where the integrals are in the sense of Cauchy principal value and where Γ is smooth at 'i', $c_{lk}^{i} = \frac{1}{2}\ \delta_{lk}$. When 'i' is at a point where the boundary is not smooth, the value of the integrals in equation (3.35) give different results and it is generally difficult to obtain a general expression in three dimensions.

Boundary equation (3.36) permits to solve the general boundary value problem of elastostatics. If displacements are known over the whole domain, equation (3.36) produces an integral equation of the first kind, if tractions are known over all the boundary an integral equation of the second kind is obtained and finally a combination of both types of boundary conditions results in a mixed integral equation.

3.5 Boundary Element Formulation

Start with the integral expression, i.e.,

$$c_{lk}^{i} u_{k}^{i} + \int_{\Gamma} u_{\mathrm{k}} p_{\mathrm{lk}}^{*}\ \mathrm{d}\Gamma\ = \int_{\Gamma} p_{\mathrm{k}} u_{\mathrm{lk}}^{*}\, \mathrm{d}\Gamma + \int_{\Omega} b_{\mathrm{k}} u_{\mathrm{lk}}^{*}\, \mathrm{d}\Omega \qquad 3.37$$

where the fundamental solution has been assumed to satisfy the following equation,

$$\sigma_{jk,j}^{*} + \Delta^{i} = 0 \qquad 3.38$$

Equation (3.37) applies for points on the boundary or internal points (with $c_{lk}^{i} = \delta_{lk}$). Smooth surfaces give $c_{lk}^{i} = \frac{1}{2}\delta_{lk}$ and corners produce a different type of c_{lk}^{i} array as will be discussed shortly.

The fundamental solution for an isotropic material in plane strain,

$$u_{lk}^{*} = \frac{1}{8\pi G(1-\nu)}\left[(3-4\nu)\ln\left(\frac{1}{r}\right)\delta_{lk} + \frac{\partial r}{\partial x_{l}}\frac{\partial r}{\partial x_{k}}\right]$$

$$\mathrm{P}_{\mathrm{lk}}^{*} = -\frac{1}{4\pi(1-\nu)\mathrm{r}}\ \left[\frac{\partial \mathrm{r}}{\partial \mathrm{n}}\left\{(1-2\nu)\delta_{\mathrm{kl}} + 2\frac{\partial \mathrm{r}}{\partial \mathrm{x_k}}\frac{\partial \mathrm{r}}{\partial \mathrm{x_l}}\right\} - (1-2\nu)\left(\frac{\partial \mathrm{r}}{\partial \mathrm{x_l}}\mathrm{n_k} - \frac{\partial \mathrm{r}}{\partial \mathrm{x_k}}\mathrm{n_l}\right)\right] \qquad 3.39$$

p_{lk}^{*} and u_{lk}^{*} represent the tractions and displacements in the k direction due to a unit load in the l direction acting at 'i'.

One can now write equation (3.37) in matrix form by defining the following arrays. The fundamental solution components can be written as two 2 x 2 matrices with elements u_{lk}^{*} and p_{lk}^{*}, i.e.,

$$u^* = \begin{bmatrix} u_{11}^* & u_{12}^* \\ u_{21}^* & u_{22}^* \end{bmatrix}; \qquad p^* = \begin{bmatrix} p_{11}^* & p_{12}^* \\ p_{21}^* & p_{22}^* \end{bmatrix} \tag{3.40}$$

The displacement, tractions and body forces vectors are

$$\mathrm{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}; \qquad \mathrm{p} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}; \qquad \mathrm{b} = \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} \tag{3.41}$$

Hence the basic equation becomes,

$$c^i u^i + \int_\Gamma p^* \mathrm{u}\ d\Gamma = \int_\Gamma u^* \mathrm{p}\ d\Gamma + \int_\Omega u^* \mathrm{b}\ d\Omega \tag{3.42}$$

where the u^i defines the displacements at the internal or boundary point i

where the load is applied. c^i is a 2 x 2 array of constant which values depend on the type of point under consideration. If ‘i’ is an internal point

$$c^i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.43}$$

If ‘i’ is a boundary point on a smooth surface then,

$$c^i = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \tag{3.44}$$

If ‘i’ is a corner, we will have

$$c^i = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \tag{3.45}$$

where the c_{lk} values will depend on the type of corner under consideration as we will see shortly.

3.6 Constant Element Formulation

Consider now that the surface of the boundary under study is discretized using constant elements (figure 3.4). This implies that the values of u and p are assumed to be constant on each element and equal to the value at the mid-node of the element. One can also discretize the interiors of the domain in a number of cells which are required for the integration of the body force term in (3.42). These cells are used only for numerical integration of the body force terms and in certain cases, they can be taken to the boundary. Consider here that the body **Ω** is discretized into N boundary elements and M internal cells, hence formula (3.42) can be written as,

$$c^i u^i + \sum_{j=1}^{N} \left\{ \int_{\Gamma_j} p^* \mathrm{d}\Gamma \right\} u^j = \sum_{j=1}^{N} \left\{ \int_{\Gamma_j} u^* \mathrm{d}\Gamma \right\} p^j + \sum_{s=1}^{M} \left\{ \int_{\Omega_s} u^* \mathrm{b}\, \mathrm{d}\Omega \right\} \quad 3.46$$

The above equation corresponds to the particular node 'i' where the unit forces are assumed to be acting.

Notice that terms such as $\int_{\Gamma_j} u\, \mathrm{d}\Gamma$ and $\int_{\Gamma_j} p\, \mathrm{d}\Gamma$ relate node 'i' to the element or node 'j'. They produce a type of influence coefficient. After integration the integrals produce two 2 x 2 sub-matrices called $\hat{H}^{ij}$ and G^{ij}. Numerical integration of the body forces term can be carried out as follows,

$$\sum_{s=1}^{M} \left\{ \int_{\Omega_s} u^* \mathrm{b}\, \mathrm{d}\Omega \right\} = \sum_{s=1}^{M} \left\{ \sum_{p=1}^{t} (u^* b)_p w_p \right\} \Omega_s = \sum_{s=1}^{M} B^{is} \quad 3.47$$

Notice that this produces two components of B^{is} i.e. B_1^{is} and B_2^{is} after the numerical integration has taken place. w_p are the weighting coefficients and Ω_s the area of the cell under consideration. The function (u*b) has to be calculated at the p integration points, where p varies from 1 to r.

Equation (3.46) can now be written

$$\mathrm{c}^i \mathrm{u}^i + \sum_{j=1}^{N} \hat{\mathrm{A}}^{ij} \mathrm{u}^j = \sum_{j=1}^{N} \mathrm{G}^{ij} \mathrm{p}^j + \sum_{s=1}^{M} \mathrm{B}^{is} \quad 3.48$$

This equation relates the values of u at node 'i' with the values of u's and p's at all the nodes on the boundary, including 'i'. Notice that in this case smooth boundary - c^i is a 2 x 2 matrix with only $1/2$ on the diagonal.

Equation (3.48) can be written in a more compact manner if we define

$$\mathrm{H}^{ij} = \hat{H}^{ij} \qquad \text{when } i \neq j$$

$$\mathrm{H}^{ij} = \hat{H}^{ij} + c^i \qquad \text{when } i = j \quad 3.49$$

Formula (3.48) has then the following form

$$\sum_{j=1}^{N} \mathrm{H}^{ij} \mathrm{u}^j = \sum_{j=1}^{N} \mathrm{G}^{ij} \mathrm{p}^j + \sum_{s=1}^{M} \mathrm{B}^{is} \quad 3.50$$

If one applies (3.50) to all boundary points the result can also be written in matrix form, i.e.

$$\mathbf{HU} = \mathbf{GP} + \mathbf{B} \quad 3.51$$

where **H** and **G** are 2N x 2N matrices (N number of boundary nodes).

Equation (3.51) has to be rearranged when applying the boundary conditions. The process consists of moving to the left-hand side all columns multiplied by an unknown and accumulating on the right-hand side vector **F** all the values obtained by multiplying the known boundary conditions by the terms in the corresponding columns. This produces the following system of equations:

$$\mathbf{AX} = \mathbf{F} + \mathbf{B} \qquad 3.52$$

The vector **X** represents all unknowns - displacements or tractions - in the problem. Once (3.52) is solved all boundary values are found.

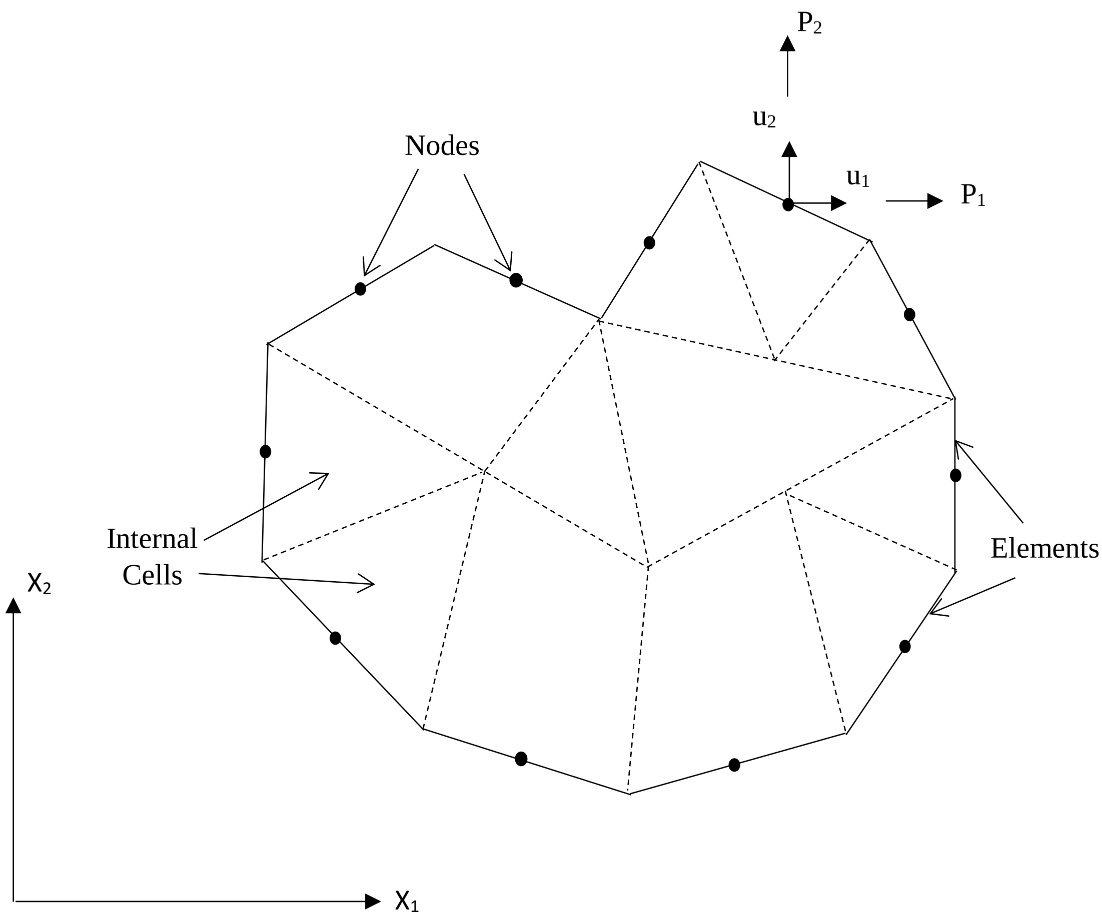


Figure 3.4 Two-dimensional body divided into boundary elements and internal cells

3.7 Results at Internal Points

Once the values of displacements and tractions are known on the boundary it is possible to calculate the displacements and stresses at any interior point. The displacements are given by formula (3.46) with $c^i = I$ (I is unit diagonal matrix), i.e.

$$\mathrm{u}^i = \int_\Gamma \mathrm{u}^*\mathrm{p}d\Gamma - \int_\Gamma \mathrm{p}^*\mathrm{u}d\Gamma + \int_\Omega \mathrm{u}^*\mathrm{b}d\Omega \tag{3.53}$$

This expression can be discretized

$$\mathrm{u}^i = \sum_{j=1}^{N} \left\{\int_{\Gamma_j} \mathrm{u}^* d\Gamma\right\} \mathrm{p}^j - \sum_{j=1}^{N} \left\{\int_{\Gamma_j} \mathrm{p}^* d\Gamma\right\} \mathrm{u}^j + \sum_{s=1}^{M} \left\{\int_{\Omega_s} \mathrm{u}^*\mathrm{b}d\Omega\right\} \tag{3.54}$$

and then integrated numerically, analytically or by a combination of both techniques.

The internal stresses can be found using the formula,

$$\sigma_{kl} = \int_\Gamma \mathrm{D}_{kl}\mathrm{p}d\Gamma - \int_\Gamma \mathrm{S}_{kl}\mathrm{u}d\Gamma + \int_\Omega \mathrm{D}_{kl}\mathrm{b}d\Omega \tag{3.55}$$

where

$$\mathrm{D}_{kl} = [D_{1kl}, D_{2kl}]$$
$$\mathrm{S}_{kl} = [S_{1kl}, S_{2kl}]$$

Formula (3.55) can be written in discretized form as follow

$$\sigma_{kl} = \sum_{j=1}^{N} \left\{\int_{\Gamma_j} \mathrm{D}_{kl} d\Gamma\right\} \mathrm{p}^j - \sum_{j=1}^{N} \left\{\int_{\Gamma_j} \mathrm{S}_{kl} d\Gamma\right\} \mathrm{u}^j + \sum_{s=1}^{M} \left\{\int_{\Omega_s} \mathrm{D}_{kl}\mathrm{b}d\Omega\right\} \tag{3.56}$$

3.8 Integration

As we will see for the case of the quadratic elements, all integrals in the above expressions can be done numerically. For the case of constant elements however it is simpler and more exact to carry out some integrations analytically, particularly those over the element with the singularity, i.e. for the case i = j. Everywhere else the values of the integrals in H^{ij} and G^{ij} have been computed using a four-point Gauss quadrature formula. Notice that values of the sub-matrices H^{ij} (for i = j) are easy to calculate using rigid body considerations. The terms in G^{ii} are the only coefficients to be computed analytically. The integrals are carried out for all the elements in G^{ii}, i.e.

$$G^{ii} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}^{ii} \quad 3.57$$

Substituting the fundamental solution into the corresponding integrals one can define three terms as,

$$\mathrm{G}_{11} = \frac{1}{8\pi\mu(1-v)}\left[(3-4v)\int_{\Gamma_i} \ln\left(\frac{1}{r}\right) d\Gamma + \int_{\Gamma_i} \left(\frac{\partial r}{\partial x_1}\right)^2 d\Gamma\right] \quad 3.58$$

$$\mathrm{G}_{12} = \mathrm{G}_{21} = \frac{1}{8\pi\mu(1-v)}\left[\int_{\Gamma_i} \frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_2} d\Gamma\right] \quad 3.59$$

$$\mathrm{G}_{22} = \frac{1}{8\pi\mu(1-v)}\left[(3-4v)\int_{\Gamma_i} \ln\left(\frac{1}{r}\right) d\Gamma + \int_{\Gamma_i} \left(\frac{\partial r}{\partial x_2}\right)^2 d\Gamma\right] \quad 3.60$$

Notice that Γ_i refer to element 'i' over which the singularity is acting. The derivatives of r in formulae (3.58) to (3.60) are given, for the general case, by (figure 3.5(a)).

$$\frac{\partial r}{\partial x_1} = \frac{r_1}{r} = \frac{x_1^k - x_1^i}{|r|}$$

$$\frac{\partial r}{\partial x_2} = \frac{r_2}{r} = \frac{x_2^k - x_2^i}{|r|} \quad 3.61$$

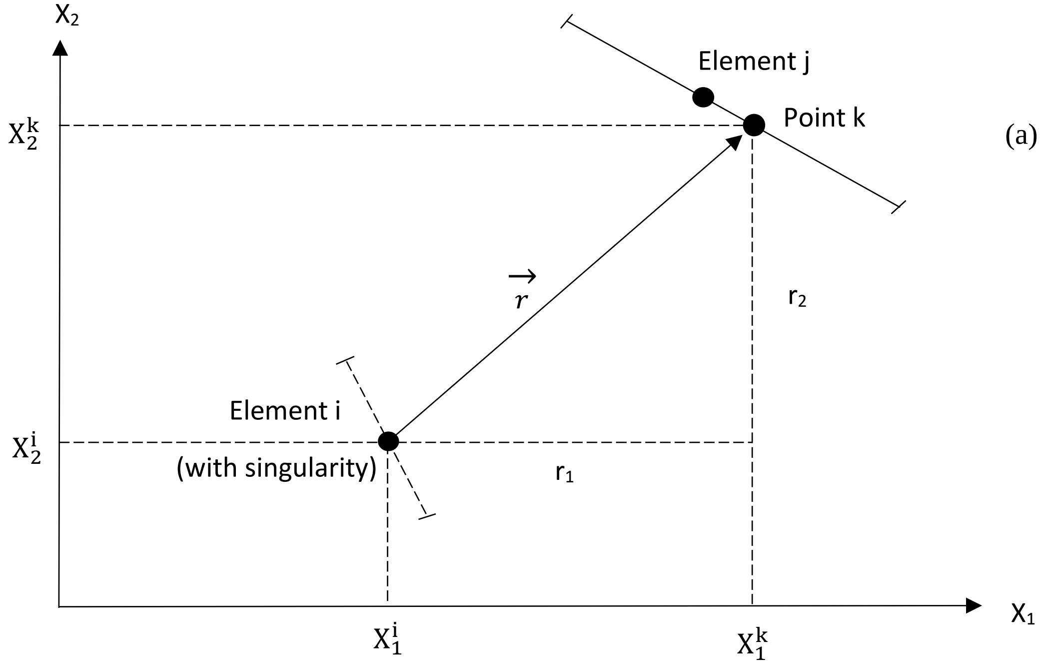
where $|r|$ is the magnitude of the distance vector $\vec{r}$.

With the above definitions, one can now concentrate on the case for which the integration is carried out over the element 'i' which contains the singularity (figure 3.5(b)). Let us consider that the element starts at extreme point (1) and finishes at extreme point (2) shown in figure 3.5(a). The distances from these points to the singularity located at i in the centre of the element is R, while small r is the variable distance from i to any point over the element, i.e. r is equivalent to Γ.

Using the θ - r system defined in figure 3.5(b) one finds the following relationship for (3.61)

$$\frac{\partial r}{\partial x_1} = \frac{r_1}{r} = \cos\theta$$

$$\frac{\partial r}{\partial x_2} = \frac{r_2}{r} = \sin\theta \quad 3.62$$



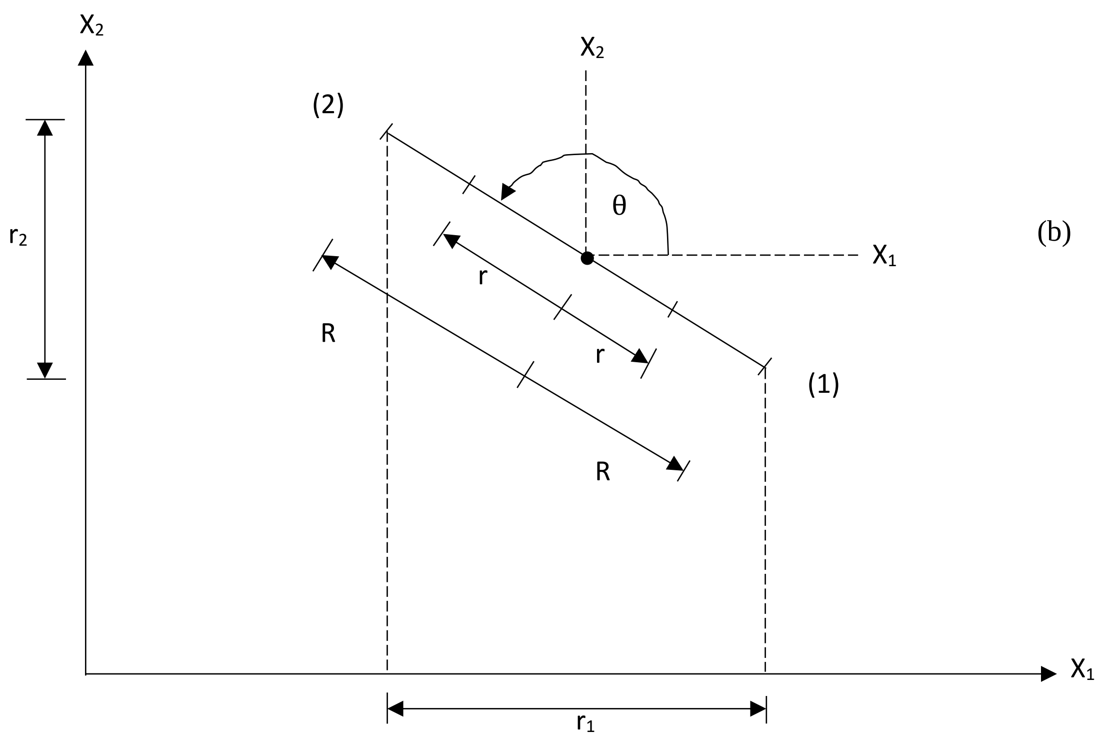


Figure 3.5 (a) Geometrical definition, (b) Definition for integration over collocation element

One can now write formulae (3.58) to (3.60) taking limits around the singularity.

$$G_{11} = \lim_{\in\to 0}\left\{\frac{1}{8\pi\mu(1-\nu)}\left[2(3-4\nu)\int_{\epsilon}^{R}\ln\frac{1}{r}\; dr + 2\int_{\epsilon}^{R}(\cos\theta)^2\; dr\right]\right\}$$

$$= \lim_{\in\to 0}\left\{\frac{1}{8\pi\mu(1-\nu)}\left[2(3-4\nu)\left\{\left[r\ln\frac{1}{r}\right]_{\epsilon}^{R} + \;[r]_{\epsilon}^{R}\right\} + 2R\,(\cos\theta)^2\right]\right\}$$

$$= \frac{2R}{8\pi\mu(1-\nu)}\left[(3-4\nu)(1-\ln R) + \frac{(r_1)^2}{4R^2}\right] \qquad 3.63$$

$$G_{11} = G_{21} = \lim_{\in\to 0}\left\{\frac{1}{8\pi\mu(1-\nu)}\left[2\int_{\epsilon}^{R}\sin\theta\cos\theta\; d\Gamma\right]\right\}$$

$$= \frac{1}{8\pi\mu(1-\nu)}[2R\sin\theta\cos\theta] = \frac{2R}{8\pi\mu(1-\nu)}\left[\frac{r_1 r_2}{4R^2}\right] \qquad 3.64$$

$$G_{22} = \lim_{\in\to 0}\left\{\frac{1}{8\pi\mu(1-\nu)}\left[2(3-4\nu)\int_{\epsilon}^{R}\ln\frac{1}{r}\; dr + 2\int_{\epsilon}^{R}(\sin\theta)^2\; dr\right]\right\}$$

$$= \lim_{\in\to 0}\left\{\frac{1}{8\pi\mu(1-\nu)}\left[2(3-4\nu)\left\{\left[r\ln\frac{1}{r}\right]_{\epsilon}^{R} + \;[r]_{\epsilon}^{R}\right\} + 2R(\sin\theta)^2\right]\right\}$$

$$= \frac{2R}{8\pi\mu(1-\nu)}\left[(3-4\nu)(1-\ln R) + \frac{(r_2)^2}{4R^2}\right] \qquad 3.65$$

CHAPTER 4

RESULTS and DISCUSSIONS

Case Study 1 (Validation Problem)

Consider a very elementary **plane stress problem** of a square plate subjected to a uniform tensile load along the x-direction. The geometry and the boundary element discretization are shown in fig. 4.1.

The data for the sample problem (in consistent units) are: -

Size of the plate		$= 10 \times 10$
Thickness of the plate		$= 1$
Modulus of elasticity,	E	$= 100$
Poisson's ratio,	ν	$= 0.2$
The loading,	p_x	$= 1$
Number of elements		$= 4$
Number of nodes		$= 8$
Number of interior nodes		$= 1$

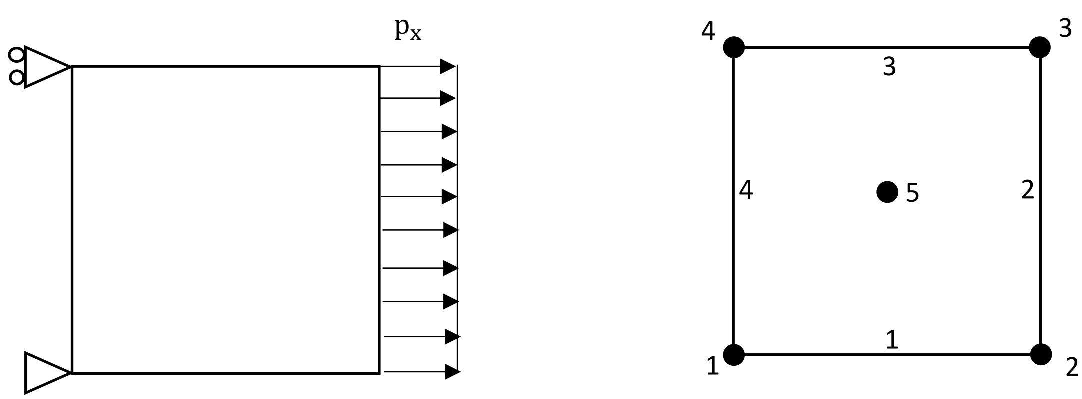


Figure 4.1 Geometry of the sample problem and its boundary element discretization into linear elements.

Hand Calculation

Displacement, $\delta = {(P_x A)}/{(EL)}$

Here, E = 100, L = 10, P_x = (1 x 10) = 10, A = (10 x 1) = 10.

Hence,

$$\delta = {(P_x A)}/{(EL)} = \frac{(10 \times 10)}{(100 \times 10)} = \frac{1}{10} = 0.1 \text{ unit.}$$

Boundary Element Method

Using constant boundary element formulation, we get end displacement = 0.1 unit

Table 4.1: Output for the numerical problem

	Number of elements			
	4	8	16	40
End Displacement	0.1008	0.10032	0.10053	0.10006

From the above discussion, it is clearly evident that as we define the number of boundary elements from 4 to 40, the solution monotonically converged to the exact solution. So, our numerical formulation is validated. This is a **plane stress** problem, where modified E and υ values were used.

Case Study 2 (Validation Problem)

Figure 4.2 describes a circular cavity under internal pressure in an **infinite medium** (plane strain problem). Numerical results will be compared against the known analytical solution, using constant elements. The boundary is divided here into 6 constant elements and 5 internal nodes are defined at which displacements and stresses will be found.

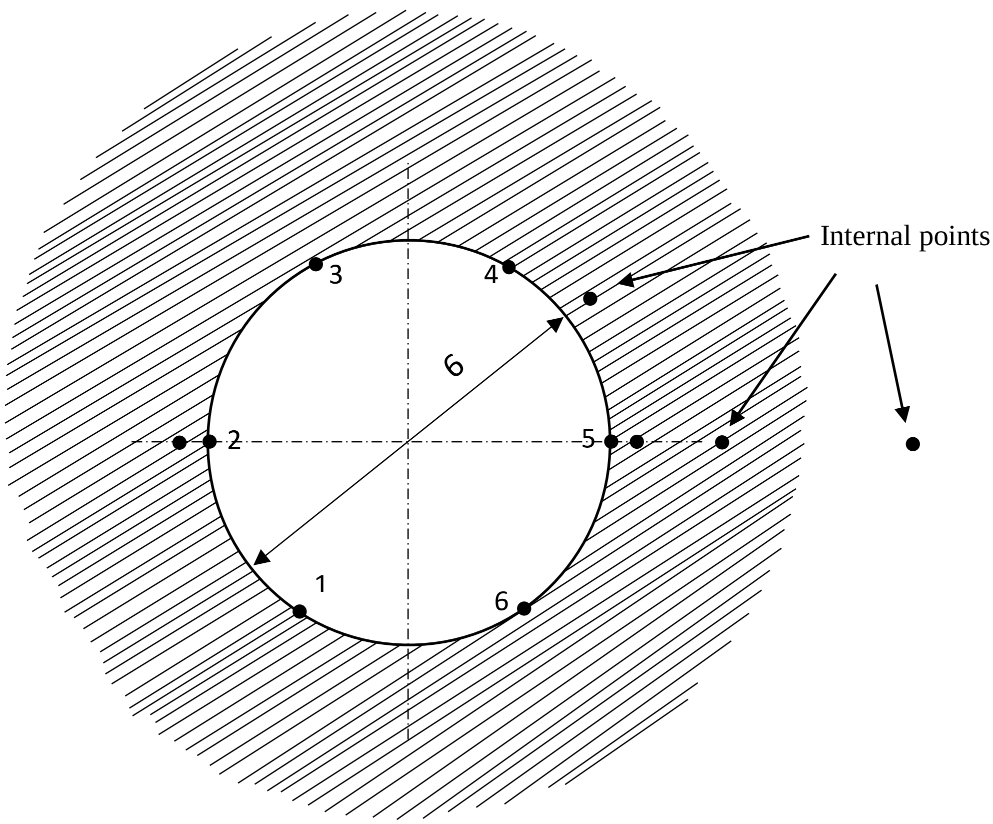


Figure 4.2 Circular cavity under internal pressure boundary element mesh and internal point description

Table 4.2: Output for the case study 2 and convergence study

Distance from centre of cavity	Numerical values with N elements/side						Analytical results **[Ref. 17]**
	N = 6	N = 12	N = 24	N = 48	N = 72	N = 96	
4	-30.326	-62.053	-57.234	-56.642	-56.52	-56.46	-56.25
6	-21.978	-25.454	-25.295	-25.174	-25.12	-25.09	-25
10	-8.8279	-9.1366	-9.106	-9.0627	-9.043	-9.03	-9

Table 4.3: Error and Convergence study for increasing number of elements

Distance from centre of cavity	% Error for Numerical values with N elements/side					
	N = 6	N = 12	N = 24	N = 48	N = 72	N = 96
4	-46.09	10.32	1.75	0.697	0.48	0.373
6	-12.09	1.82	1.18	0.696	0.48	0.36
10	-1.91	1.52	1.18	0.697	0.478	0.333

Discussions

Number of the boundary elements has been increased from six to ninety-six. From the results in Tables 4.2 and 4.3 it could be seen that with increasing the number of boundary elements, numerical results of interior nodes approach to analytical results. In case of 96 boundary elements solution is almost equal to exact analytical solution. So, increasing the number of meshes, numerical solution converges with analytical solution.

Case Study 3

Figure 4.3 describes two circular cavities under internal pressure in an infinite medium (plane strain problem). The boundary is divided into 12 constant elements and 3 internal nodes are defined at which displacements and stresses are computed.

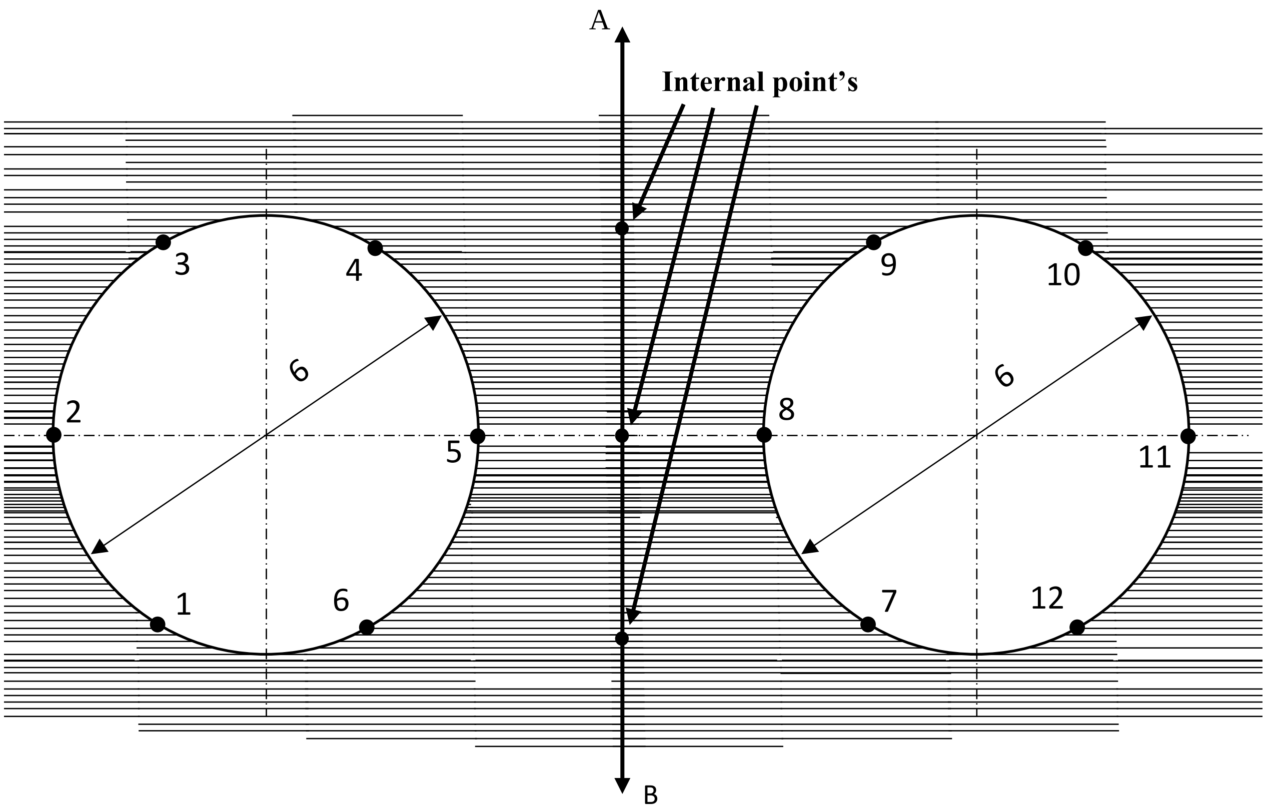


Figure 4.3 Two circular cavity under internal pressure boundary element mesh and internal point description

Table 4.4a: Output value for internal point's displacement in X-direction

Internal points	Numerical values of displacement with N elements/side					
	N = 12	N = 24	N = 48	N = 96	N = 144	N = 192
10, 0	-1.02E-10	7.05E-10	1.90E-09	7.30E-08	-1.88E-08	1.21E-08
10, 5	-3.09E-10	6.19E-10	1.96E-09	7.73E-08	-1.95E-08	1.24E-08
10, -5	-1.46E-10	5.67E-10	2.73E-09	7.73E-08	-1.95E-08	1.28E-08

Table 4.4b: Output value for internal point's displacement in Y-direction

Internal points	Numerical values of displacement with N elements/side					
	N = 12	N = 24	N = 48	N = 96	N = 144	N = 192
10, 0	8.42E-11	-1.23E-10	1.24E-09	7.50E-10	-1.06E-08	4.37E-09
10, 5	4.02E-04	4.05E-04	4.02E-04	3.99E-04	3.97E-04	3.96E-04
10, -5	-4.02E-04	-4.05E-04	-4.02E-04	-3.99E-04	-3.97E-04	-3.96E-04

Table 4.5a: Output value for internal point's stress in X-direction

Internal points	Numerical values of σ_x with N elements/side					
	N = 12	N = 24	N = 48	N = 96	N = 144	N = 192
10, 0	-17.8128	-18.1016	-17.7382	-17.4464	-17.3199	-17.2464
10, 5	-10.3644	-10.1226	-9.8557	-9.6201	-9.5103	-9.4436
10, -5	-10.3644	-10.1226	-9.8557	-9.6201	-9.5103	-9.4436

Table 4.5b: Output value for internal point's stress in Y-direction

Internal points	Numerical values of σ_y with N elements/side					
	N = 12	N = 24	N = 48	N = 96	N = 144	N = 192
10, 0	18.3738	18.7394	18.6165	18.4825	18.4231	18.3891
10, 5	9.6939	9.7558	9.6922	9.6088	9.5682	9.5437
10, -5	9.6939	9.7558	9.6922	9.6088	9.5682	9.5437

Table 4.5c: Output value for internal point's shear stress

Internal points	Numerical values of τ_{xy} with N elements/side					
	N = 12	N = 24	N = 48	N = 96	N = 144	N = 192
10, 0	-2.37E-06	1.29E-06	5.01E-07	2.28E-07	2.23E-06	4.95E-06
10, 5	-1.40E-07	-1.36E-06	1.19E-05	1.90E-04	-5.19E-05	3.46E-05
10, -5	2.37E-06	-6.19E-07	-1.14E-05	-1.93E-04	4.69E-05	-2.99E-05

Discussions

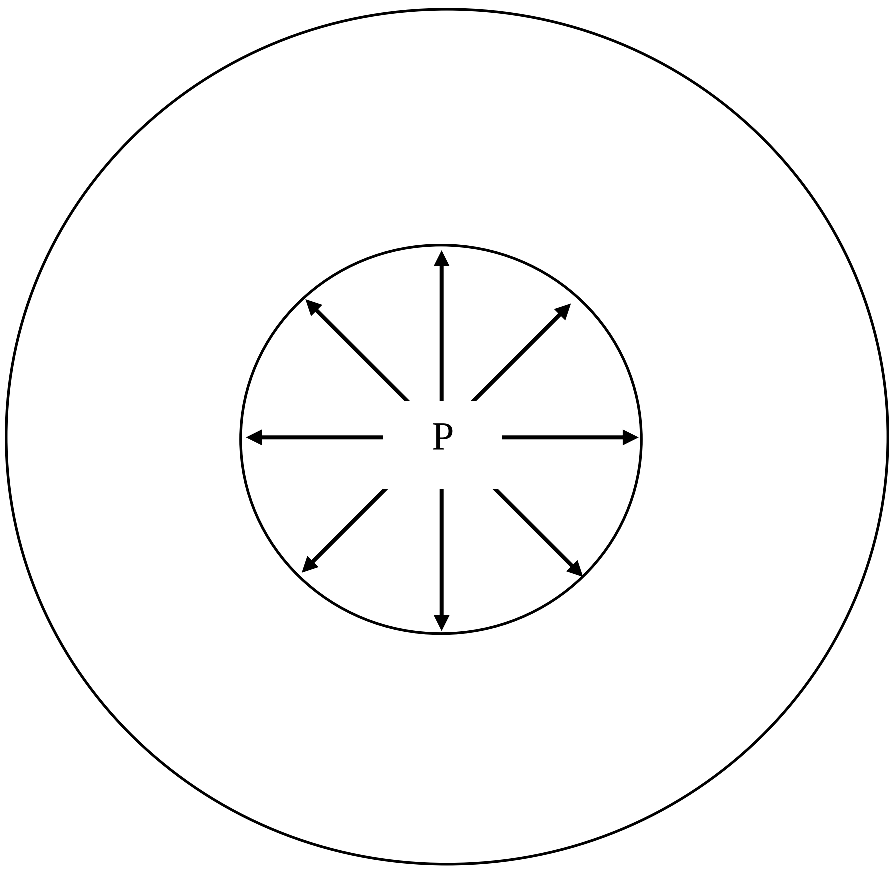
From the above results, it is observed that the displacements at points (-3,0) and (23,0) are equal but in opposite directions. Similarly, the displacements at points (3,0) and (17,0) are also equal but in opposite directions. When considering a single cavity, the displacement of (-3,0) and (3,0) will be the same. However, for two cavities, the displacement at (3,0) is comparatively smaller than that at (-3,0) due to the internal pressure incident at the second circular cavity.

Furthermore, the displacement of all points lying on the line AB is always zero due to equal pressure from both circular cavities.

Case Study 4

This finite problem represents the case of a plane strain hollow cylinder under internal pressure as shown in figure 4.4. The pressure is assumed to be p = 100 N/mm^2, while the internal and external radii are $r_1 = 10$ mm and $r_2 = 25$ mm, respectively. The elastic constants for the material are: Young's Modulus, E = 200000 N/mm^2 and Poisson's ratio, $\nu = 0.25$.

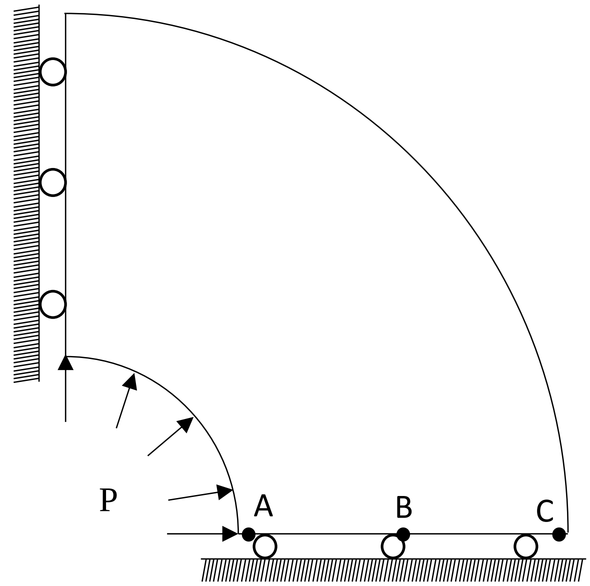
Due to the symmetry of the problem only one quarter of the section needs to be discretized. The boundary condition is shown in figure 4.4(b). Discretization consisting of 16 constant elements are used as shown in figure 4.4(c).



(a) Problem Definition

(b) Boundary Conditions

(including symmetry)



(c) 16 elements mesh
(representative drawing, all nodes are not shown)

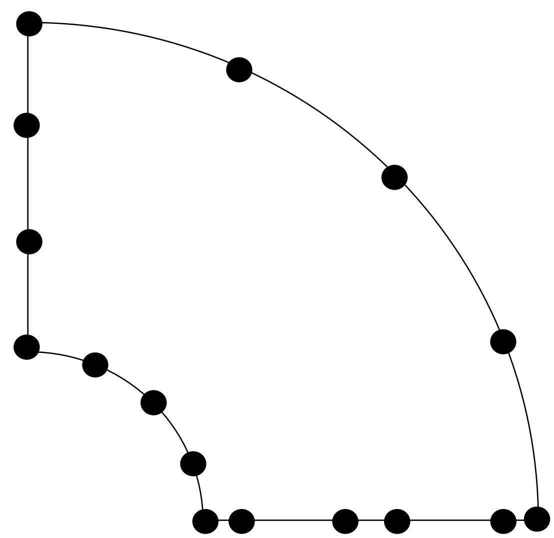


Figure 4.4 Hollow cylinder under internal pressure

Table 4.6 Radial Displacement for Hollow Cylinder under Internal Pressure (in 10^{-3} mm)

POINT	Numerical values with N elements/side							Analytical results [Ref. 17]
	N = 12	N = 20	N = 28	N = 48	N = 88	N = 108	N = 208	
A	3.7277	5.2741	6.1064	7.5184	8.0764	7.9952	7.7649	8.0325
B	5.3345	5.9628	5.9021	5.7221	5.7084	5.7250	5.7473	5.2912
C	2.6225	2.9511	3.0589	3.4500	4.4681	4.7967	5.0959	4.4526

Table 4.7 Error and Convergence study for increasing number of elements for Hollow Cylinder

POINT	% Error for Numerical values with N elements/side						
	N = 12	N = 20	N = 28	N = 48	N = 88	N = 108	N = 208
A	53.5923	34.3405	23.9788	6.4002	0.5465	0.4644	3.3315
B	0.8183	12.6928	11.5456	8.1437	7.8848	8.1985	8.6200
C	41.1018	33.7219	31.3008	22.5172	0.3481	7.7281	14.4477

Discussions

Table 4.6 shows the relevant displacements computed for the points A, B and C along the radius. The displacements are given in microns, i.e., 10^{-3} mm and are compared with the exact values. Increasing the number of meshes, results are almost equal to exact solution. In case of 88 elements, displacement at points A and C are within a range of 0.5% of the exact solution [**17**].

Case Study 5

Determine the deformation and state of stress for the pipe of figure 4.5, which is subjected to internal pressure P = 1 MPa. It has a uniform cross-section. Since it is very long in the z-direction, the produced state of stress is plane strain. The material constants are E = 2×10^5 kN/m^2 and $\nu = 0.20$. All the other data are given in Fig. 4.5. The boundaries are divided here into 100 constant elements and 292 internal points at which displacements and stresses are computed.

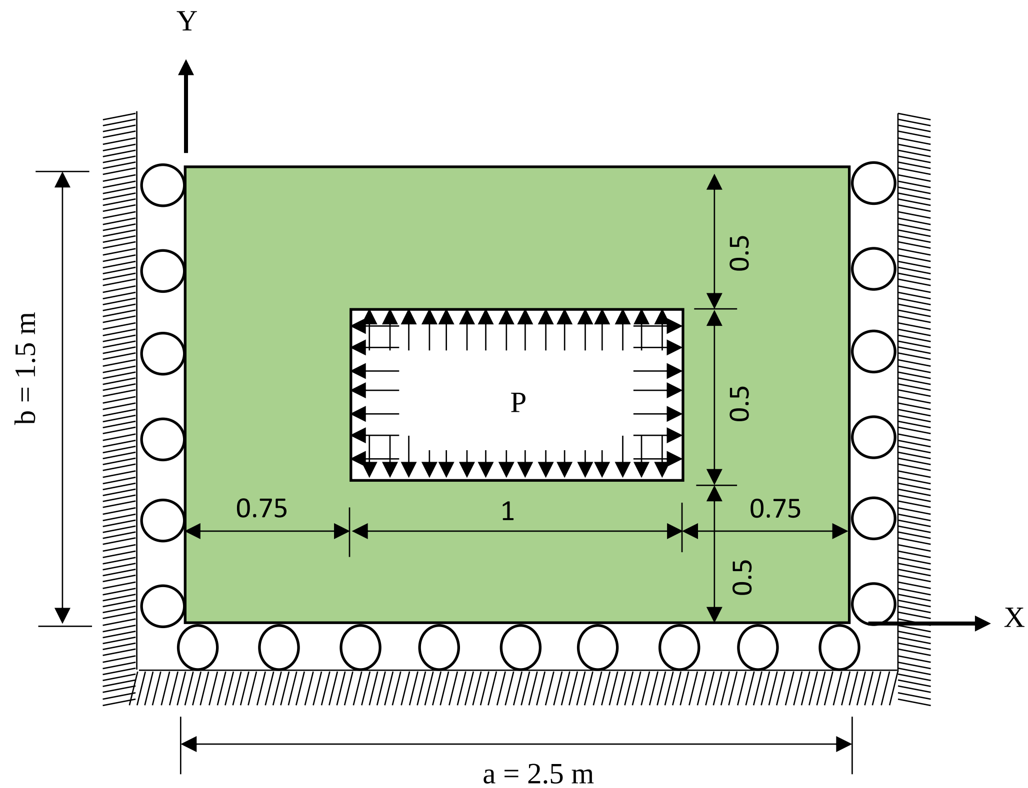


Figure 4.5 Pipe under uniform pressure

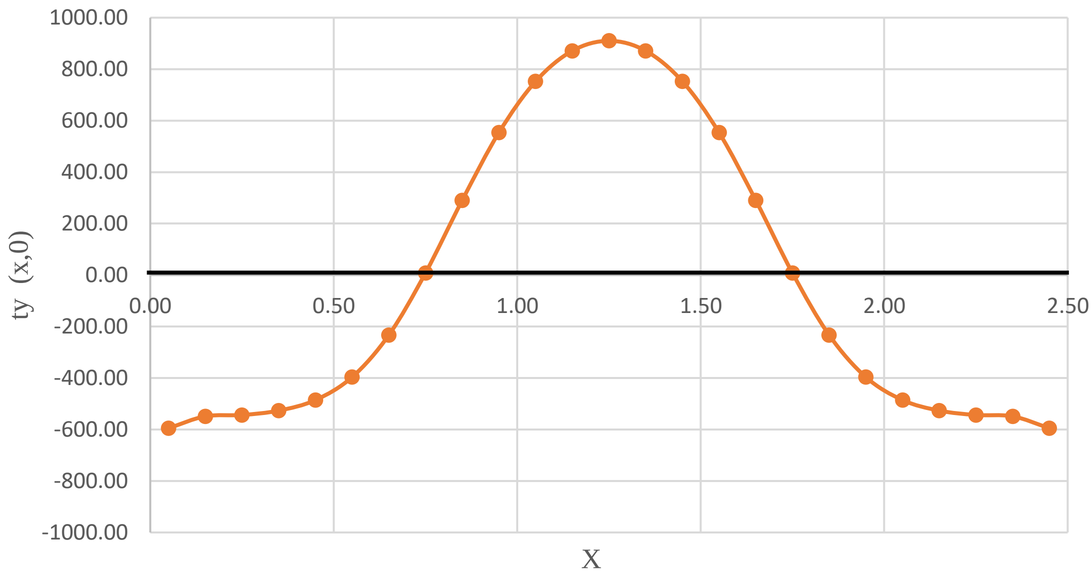


Figure 4.6 Distribution of t_y along the boundary y = 0

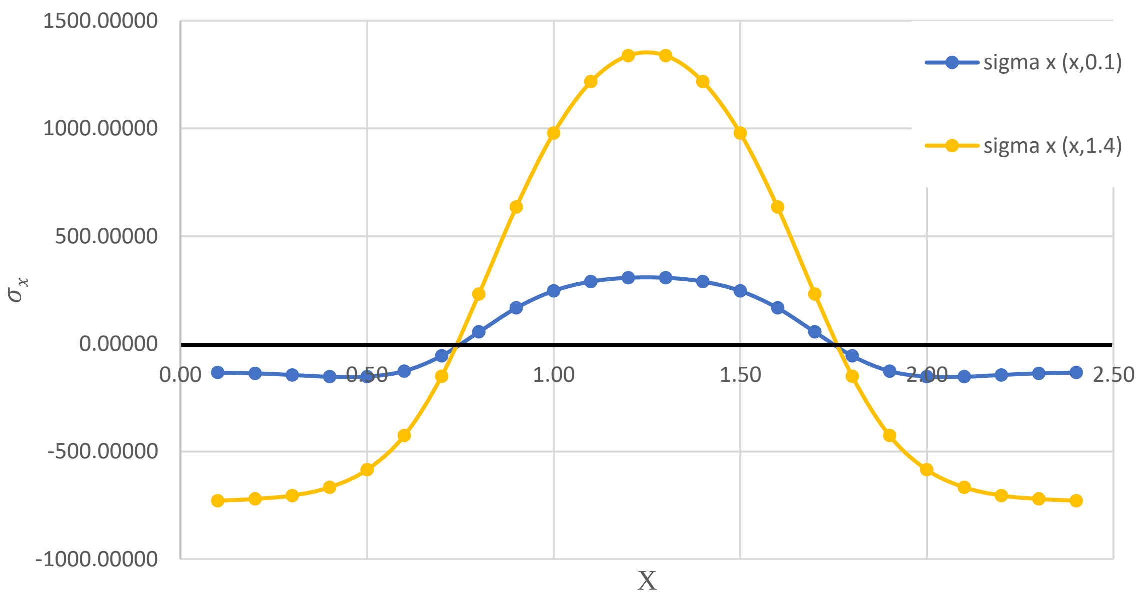


Figure 4.7 Distribution of σ_x along the lines y = 0.1and y = 1.4

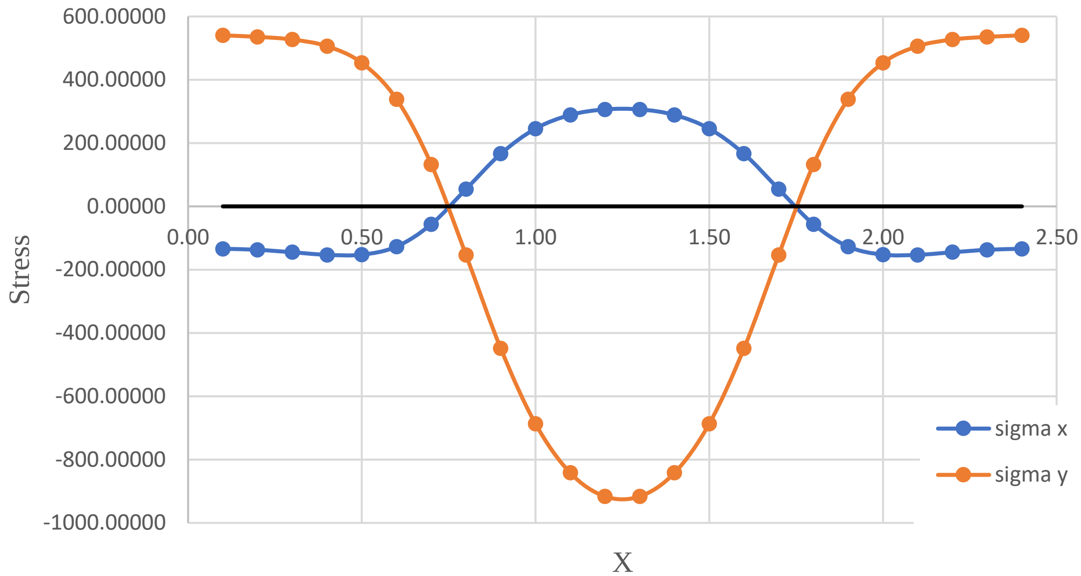


Figure 4.8 Distribution of σ_x and σ_y along the line y = 0.1

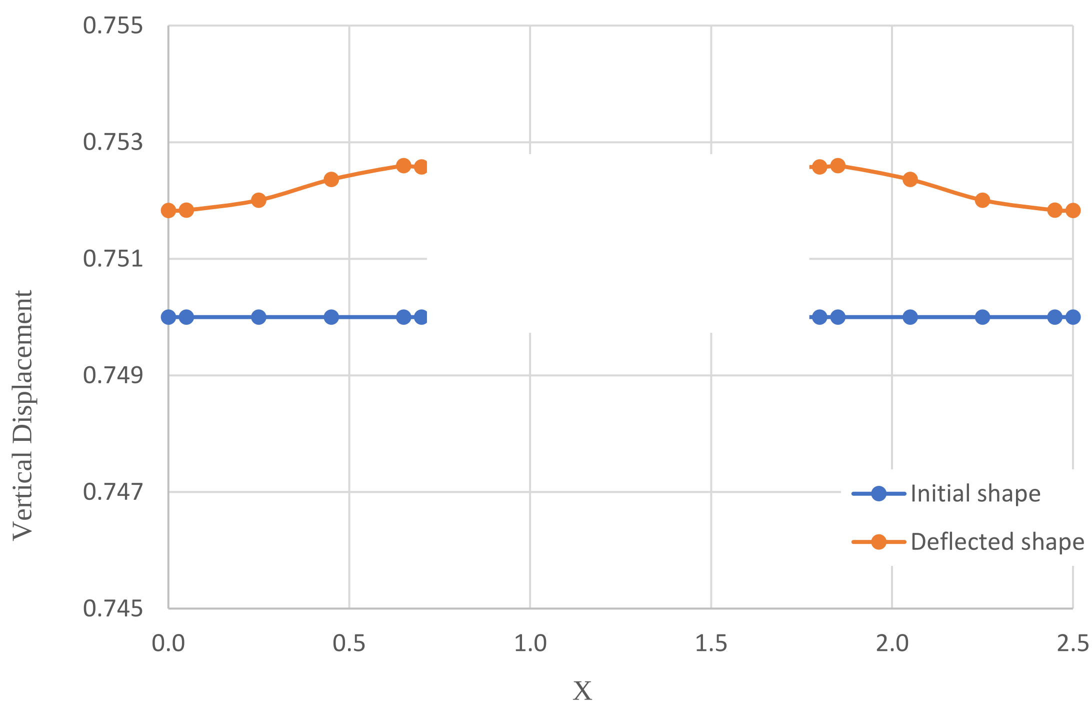


Figure 4.9 Deflection of internal points lying on the line y = 0.75

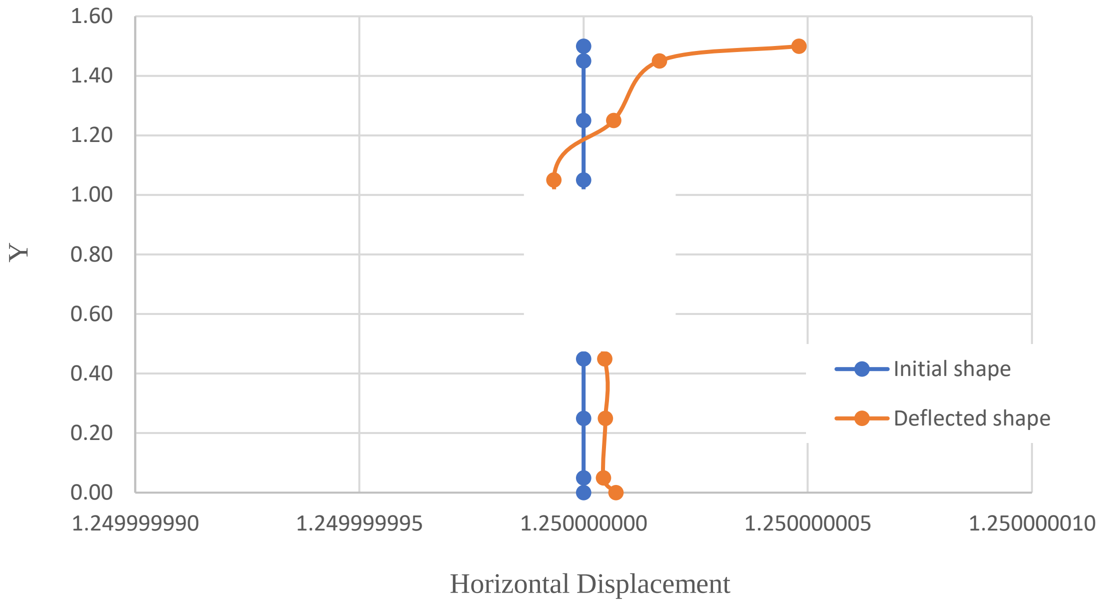


Figure 4.10 Deflection of internal points lying on the line x = 1.25

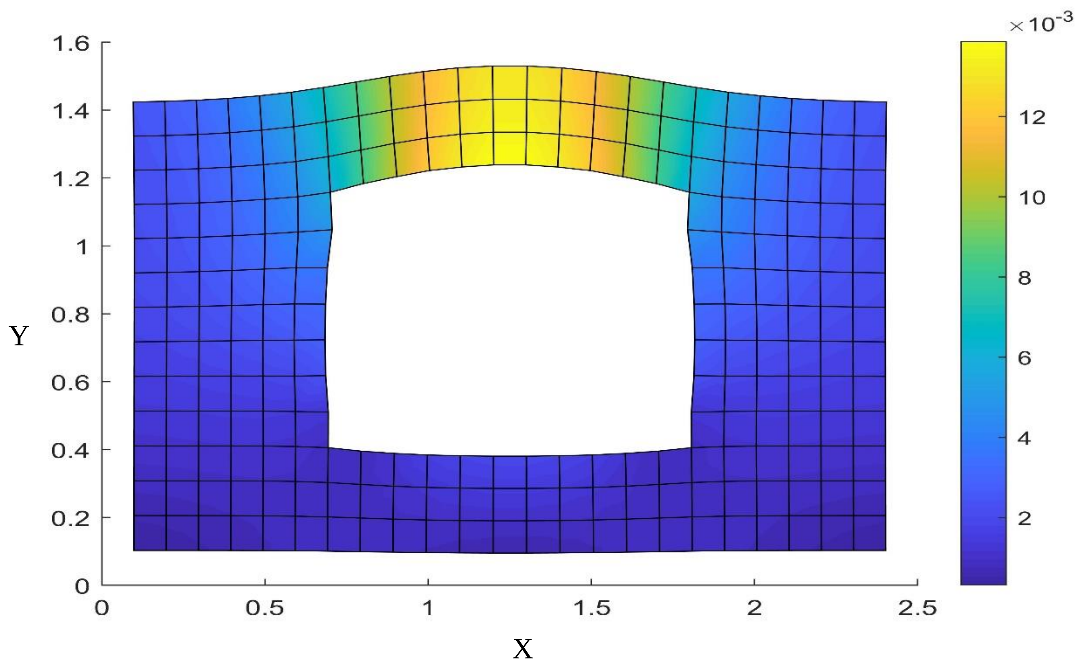


Figure 4.11 Deformed pipe cross-section

Discussions

In Fig 4.6, The traction t_y at the bottom face is equal and in opposite direction to stress at bottom because of the rigid roller support. Traction is maximum at middle portion, it decreases gradually towards both ends because pressure is applied on middle portion only. The total upward force at central portion is nullified by the downward forces on the outer flanks, there being no applied vertical forces.

In Fig 4.7, The intensity of σ_x near the free edge is more than intensity near supports. Variation with respect to distance is more at free edge. These are because of the reaction from support.

In Fig 4.8, Intensity of σ_y is more than σ_x because of the direct pressure along y direction above this line. The total positive pressure is equal to the total negative pressure along any line.

In Fig 4.9, The displacement along any horizontal line is higher towards middle portion due to higher stress near the middle portion.

In Fig 4.10, As constant elements are used, minimum deflection observed (4.80E-09 mm), for linear or quadratic elements this deflection will be zero.

CHAPTER 5

CONCLUSION

Computational approaches are necessary to address any complicated engineering problem since they take into account difficult boundary conditions and boundaries. There are many different kinds of computational method, including the boundary element method, the finite element method, and the finite difference approach. Boundary element approach, among these, requires less input data, less computing time, and less memory while demonstrating the same level of accuracy.

Here, two-dimensional elastostatic problems have been solved for validation. In the case of boundary element analysis, the internal displacements are quite close to analytical results. Eventually, the same program is used to solve a plane stress problem, and the results are compared to a hand-calculated solution of the same problem and a mesh converge study performed using BEA. Next, finer meshes have been used to conduct a mesh convergence study and observed that the results manifest a monotonic convergence.

The Program is capable of solving finite and infinite two-dimensional plane stress and plane strain problems.

Results show that though we have employed constant boundary elements, the domain stresses are very closely reproduced by the BEM in all validation problems. Displacement values are also closely reproduced at the boundary.

It can be asserted that the boundary element method offers definite advantages over domain-type techniques for many problems, and its efficiency is prolific for problems with infinite boundaries.

<u>Future Scope of Study</u>

The present work may be extended by implementing linear and quadratic elements instead of constant elements. This will enable to easily define complex curved boundaries and should require smaller number of elements. Further, the two-dimensional problem may be extended to three-dimensional cases.

REFERENCES

[1] Green G. An essay on the application on mathematical analysis to the theories of electricity and magnetism, T. Wheelhouse: Nottingham; 1828.

[2] Betti E. Theoria dell' Elasticita`. Il Nuovo Cimento 1872;6-10, Ser. 2.

[3] Somigliana C. Sopra l' Equilibrio di' un Corpo Elastico Isotropo. Il Nuovo Cimento 1875;18(1):91-6 Ser. 3.

[4] Fredholm I. Sur une classe d' equations fonctionelles. Acta Mathematica 1903; 27:365-90.

[5] Jaswon MA. Integral equation methods in potential theory I. Proc R Soc Ser A 1963; 275:23-32.

[6] Symm GT. Integral equation methods in potential theory II. Proc R Soc Ser A 1963; 275:33-46.

[7] Jaswon M. and Pointer,A.R. An Integral equation solution of the Torsion Problem. Proc Roy, Soc Ser A 1963;273:237-246.

[8] Hess, 1. L., and Smith, A M. 0., Calculation of potential flow about arbitrary bodies, Progress in Aeronautical Sciences Vol. 8, (D. Kiichemann, Ed.), Pergamon Press, London, 1967.

[9] Cruse, T.A. and Rizzo F.J., A direct formulation and numerical solution of the general transient elastodynamic problem, I, J. Math. Anal. Appl. 22, 244-259 (1968).

[10] Muskhelishvili, N.1., Some Basic Problems of the Mathematical Theory of Elasticity, Nordhoff, Groningen, 1953.

[11] Mikhlin, S.G., Integral Equations, Pergamon, New York, 1957.

[12] Smirnov, V.J., Integral Equations and Partial Differential Equations, Vol IV, A Course in Higher Mathematics, Addison-Wesley, London, 1964.

[13] Kupradze, O.D., Potential Methods in the Theory of Elasticity, Daniel Davy & Co., New York, 1965.

[14] Kellogg, O.D., Foundations of Potential Theory, Dover, New York, 1953.

[15] Massonnet, C.E., Numerical use of Integral Procedures in Stress Analysis, Ed. O.C. Zienkiewicz and G.S. Holister, Wiley, London, 1966.

[16] Rizzo, F.J., An Integral Equation Approach to Boundary Value Problems of Classical Elastostatics. Quart Appl Math, Vol 25, 1967, pp 83-95.

[17] Alarcon, E., Brebbia, C. A. and Dominguez, J. The Boundary Element Method in Elasticity, Intern. Jou. of Mechanical Sciences, 20, 625-639, 1978.

[18] Brebbia, C. A. The Boundary Element Method for Engineers, Pentech Press London, 1978. Computational Mechanics Publications, Boston, 1984.

[19] Love, A. E. H. A Treatise on the Mathematical Theory of Elasticity, Dover, New York, 1944.

[20] Cruse, T. A. A Direct Formulation and Numerical Solution of the General Transient Elastodynamic Problem II, Journal of Math. Analysis and Applications, 22, 341, 1968.

[21] Barber J.R., Elasticity, Solid Mechanics and its Applications 172, 3rd ed. Springer-Verlag, New York, 2010.

[22] Wang M.Z., Xu B.X., Gao C.F., Recent General Solutions in Linear Elasticity and Their Applications, Appl. Mech. Rev., 2008, 61, 030803-1/20.

[23] Dumont NA. The consistent boundary element method for potential and elasticity: Part I – formulation and convergence theorem. EABE Eng Anal Bound Elem Methods 2023.

[24] Oliveira MFF, Dumont NA, Selvadurai APS. Boundary element formulation of axisymmetric problems for an elastic half space. Eng Anal Bound Elem 2012;26: 1478–92. http://dx.doi.org/10.1016/j.enganabound.2012.03.015.

[25] Dumont NA. The consistent boundary element method for potential and elasticity: Part II – machine-precision numerical evaluations for 2D problems. EABE Eng Anal Bound Elem Methods 2023.