Pricing Exotic Options Under Lévy Process and its Closed-form Solutions

Sudip Ratan Chandra

Faculty of Science

Jadavpur University

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by
Prof. Diganta Mukherjee, Indian Statistical Institute, Kolkata, India

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CERTIFICATE FROM THE SUPERVISOR(S)

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Diganta Mukherjee. 20/4/2023

दिगंत मुखर्जी / DIGANTA MUKHERJEE
प्राध्यापक / Professor
प्रतिचयन एवं साधिकारिक सांख्यिकी यूनिट
Sampling And Official Statistics Unit
भारतीय सांख्यिकीय संस्थान
INDIAN STATISTICAL INSTITUTE
203, बैरकपुर ट्रांक रोड, कोलकाता-700108
203, Barrackpore Trunk Road, Kol-700108

(Signature of the Supervisor(s) date with official seal)

....

Supervisor
Professor Diganta Mukherjee
Sampling and Official Statistics Unit,
Indian Statistical Institute,
203, B.T.Road, Kolkata -700108,
INDIA

....
Signature of the Supervisor

Date: ....

....

Author
Mr. Sudip Ratan Chandra
Department of Mathematics,
Faculty of Science,
Jadavpur University,
188, Raja S.C. Mallick Rd,
Kolkata - 700032,
INDIA

....
Signature of Author

Date: ....

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Abstract

Path-dependent exotic options are contracts which are traded in financial markets, characterized by payoffs which are a function of the particular continuous path that asset prices follow over the life of the relevant option. Pricing such exotic options assuming that stock price follows Jump process such as Lévy has been the main point of interest for financial analyst and researcher for many years. This is because, the stock price distribution indicate heavy tails.

A known class of Lévy process called NIG shows very close fit and therefore there is a huge interest in research community to develop an arbitrage-free pricing method for the most popular path dependent exotic options such as Asian, Barrier and Look-back with such a process. Unfortunately, there are two major difficulties in arbitrage-free pricing of exotic options with Levy process. Firstly, the derivation of the closed-form pricing expression as the distribution of the payoff function is unknown when stock price follows exponential Levy process. Secondly, the estimation of risk neutral density from the market prices through calibration methods.

The most common approach for pricing exotic options is by numerical methods such as Monte-Carlo simulation. The main reason for adopting numerical methods is that the closed-form expression is very hard to derive for options with nonlinear payoff functions under a generic class of Lévy process, especially when the distribution of payoff function is unknown. Unfortunately, the simulation methods are difficult and computationally expensive in many cases, and also quite involved. In order to address these challenges, we have proposed a novel approach for finding an arbitrage-free pricing expression (closed-form) for pricing and proposed a method based on optimal control theory to estimate risk neutral density from the market prices available.

We propose a Partial Integro Differential Equations (PIDE) and closed-form Fourier Pricing formule for several Exotic options when stock price follows exponential Lévy Process. We first develop a PIDE based on Martingale method and derive closed-form Fourier formula for pricing. The pricing expressions are simple, easy to compute and works for wide class of Lévy Processes.

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Chapter 1

Introduction

Exotic options are third generation derivatives commonly traded in various exchanges such as Chicago Board of Options Exchange (CBOE), American Stock Exchange, York Mercantile Exchange (NYMEX)1.1 and also used by fund managers from big corporations or financial institutions for hedging portfolio which are exposed to tax, accounting, legal, regulatory and other risks.

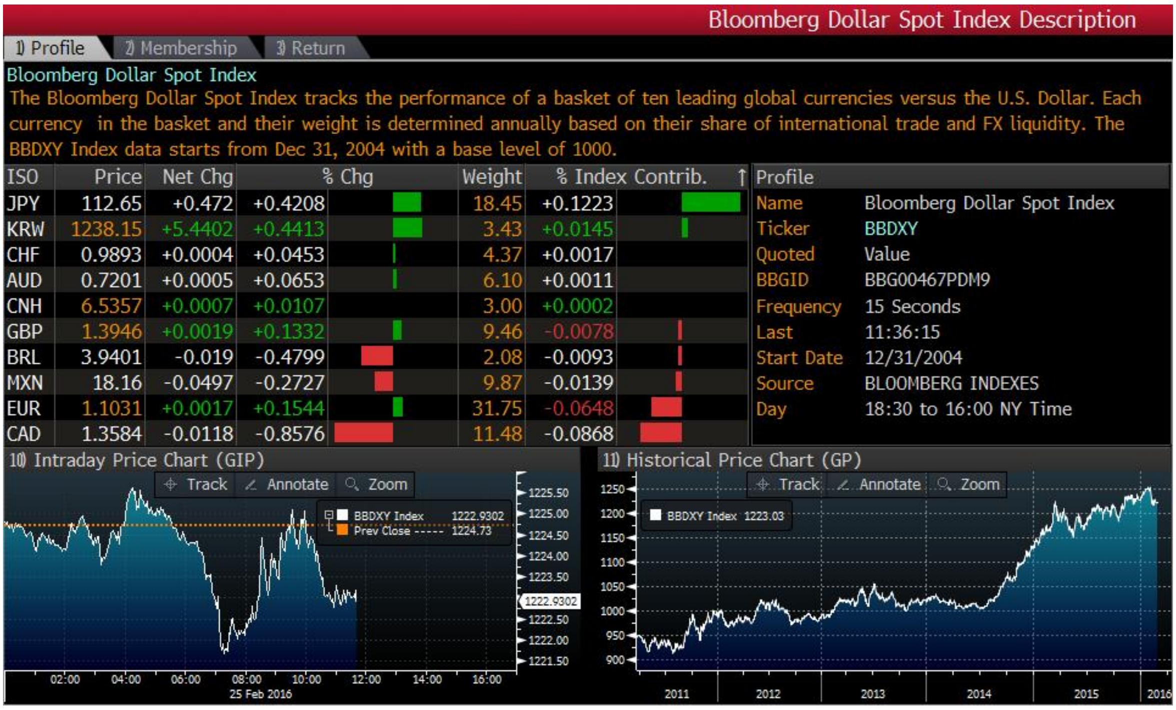

Figure 1.1: Bloomberg^{TM} Trading Terminal, www.bloomberg.com

These options can be distinguished on one or more of the following six dimensions:

1. Time-homogeneity of the structure,
2. Continuity of payoff,

3. Presence of barriers,
4. Number of assets in the structure,
5. Order of the option, and
6. Path-dependence.

and can be categorized as Path-dependent, Payoff modified, Time/Volatility-dependent and Correlation-dependent/Multifactor options.

Payoff Modified Options These are options where the payoff under the contract is modified from the conventional return, which is either zero or the difference between the strike price and the asset price at maturity. Several structures of interest exist, including:

1. Digital options, which pays a fixed amount if the underlying asset is above or below a given level at maturity of the option.
2. Contingent-premium options, which allow the linking of the option premium to be paid to the asset-price performance.
3. Power options, where the payoff under the option is an agreed multiple of the return under a conventional option.

Time/Volatility-Dependent Options These can be characterized as options where the purchaser has the right to nominate specific characteristics (e.g., type of option—put or call) as a function of time. The value of these options is particularly sensitive to volatility over a period which begins not now but in the future. These options are particularly useful when there is some event which occurs in the short term which will then potentially affect outcomes further in the future. Time/volatility-dependent structures include:

1. Chooser (or preference) options, which are not specified as either a call or a put until, at a predetermined date, the purchaser can nominate whether or not the transaction is a call or a put option.
2. Compound options, which are options on options where the holder has the right to buy or sell another predetermined option at a pre-agreed price.
3. Forward start options, which are essentially vanilla options except that the strike price is not set until some date in the future. The strike price will be set as some function of the prevailing asset price on that date.

Correlation-Dependent/Multifactor Options These typically involve a pattern of payoffs based on the relationship between multiple assets as opposed to the price, in the case of a traditional option, for single assets. A variety of structures exist, including:

1. Basket options, where the payout under the contract is related to the cumulative performance of a basket of products.
2. Exchange options, which give the purchaser the right to exchange one asset for another.
3. Quanto options, where the option contract is denominated in a currency other than that of the underlying asset to which exposure is sought or being hedged.
4. Rainbow options, where the payout is based on the relationship between multiple assets as opposed to the price of a single asset. Specific examples include outperformance (or better-of-two-assets) options and spread options.

Path-Dependent Options Path-dependent options are characterized by payoffs which are a function of the particular continuous path that asset prices follow over the life of the relevant option. The path of the underlying asset price can determine not only the payoff but also the structure of the option. Path-dependent structures include:

1. Average rate options, where the payoff upon settlement is determined by comparing the strike price with the average of the spot asset price over a specific period during the life of the option.
2. Average strike options, where the strike itself is not fixed and the payoff is determined through a comparison of the underlying price of expiration, with the strike price computed as the average of the underlying asset price over a specific period.
3. Barrier options, whereby the option contract is activated or deactivated as a function of the level of the underlying asset price. The most basic forms of barrier options are commonly known as knock-in and knock-out options.
4. One-touch options, where the payoff is a fixed amount if the underlying asset ever trades above or below a given level on any day during the lifetime of the option. A related option is the digital option.
5. Lookback options, where the purchaser has the right at expiration to set the strike price of the option at the most favorable price for the asset that has occurred during the specified time. In the case of a lookback call (put), the buyer can choose to purchase (sell) the underlying asset at the lowest (highest) price that has occurred over a specified period (typically the life of the option). Besides the floating strike variety, lookback options also come in a fixed strike variety.

The most popular and widely studied path-dependent options are Asian (average) options, lookback options and barrier options. The barrier options are cheaper than normal options due to the barrier and widely used in currency market where underlying price moves within a range. The asian option reduces the risk of regular market manipulations over entire period by damping or averaging. The lookback option is an alternative to vanila and less expensive because of price restrictions by maximum and minimum values.

The fair pricing and accurate hedging strategy of these options is extremely important to reduce loss in trading and make portfolio stable. Unfortunately, both pricing and hedging is a very difficult exercise in case of high volatility or random extreme events in the market. These extreme or tail events makes portfolio imbalanced, less diversified and more correlated, resulting large loss in trading. This is popularly known as tail risk in financial industry. Our research is on analytically pricing such exotic path-dependent options in the presence of such tail risk (in the presence of jumps).

Empirical Features of Stock Price We have studied statistical properties like mean, standard deviation, skewness and kurtosis of the log return of the stock price of major indices and displayed in Table 1.1. We have studied three datasets of Google US, LLY and MSFT US Equity[1.2,1.3] contains all daily log returns over the period 2/11/2014 to 05/27/2014 and computed statistical properties of distribution like mean, variance, skewness and Kurtosis to understand the nature of distribution in Table 1.1.

Figure 1.2: Google US Equity Price from Bloomberg between 2/11/2014 to 05/27/2014.

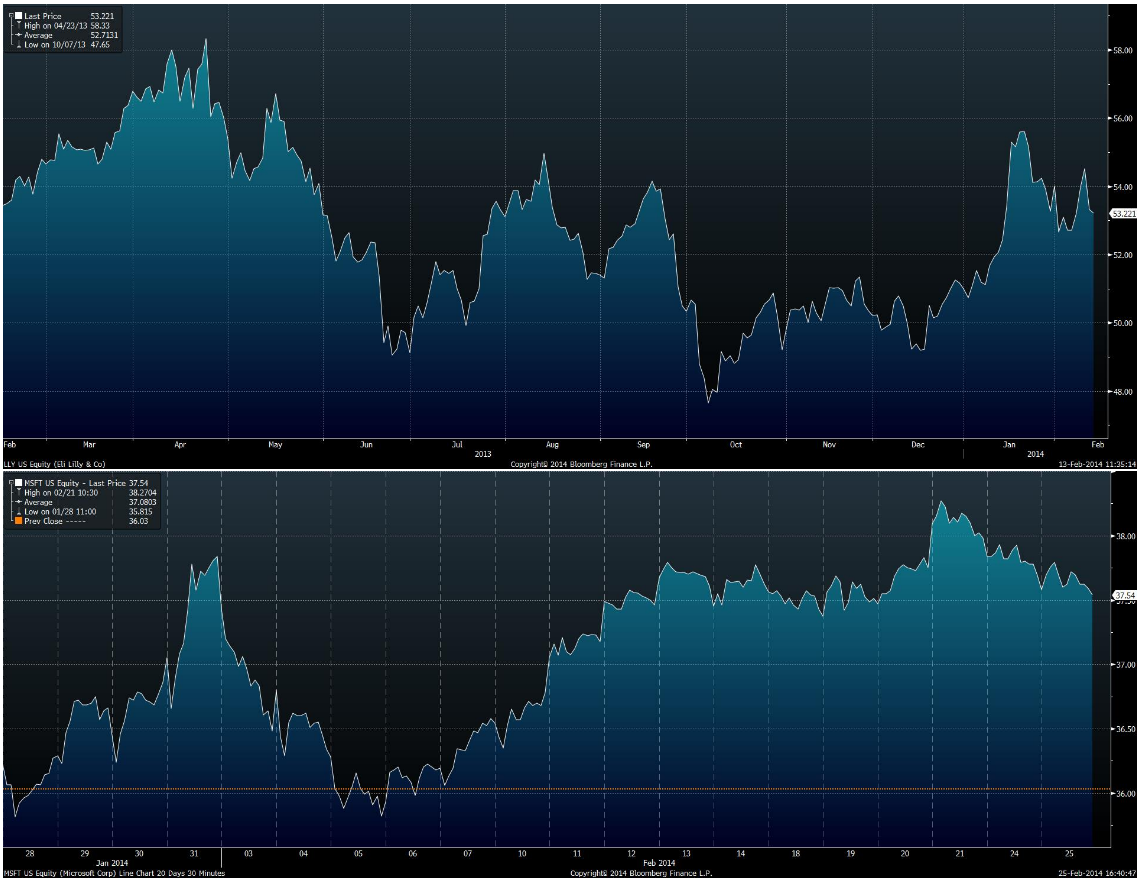

Figure 1.3: LLY and MSFT US Equity Price from Bloomberg between 2/11/2014 to 05/27/2014.

The empirical distribution of daily log returns of different indices are studied in Table 1.1. Asymmetry and fat tails are present in all three cases. The quantile plots in 1.4 indicate heavy tails in the distribution of the residuals. The JB Test and KS test confirms the non-normal distribution by rejecting normality hypothesis ($p < 0.05$).

Table 1.1: Log of Stock price and statistical properties

Index	Mean	Std	Skewness	Kurtosis	JB(p-Value)	KS(p-Value)
Google US Equity	-0.1755	0.0126	-0.3282	1.7459	0.0142	2.4162e-028
MSFT US Equity	0.0432	0.0032	-0.6161	2.2284	0.0127	2.0952e-024
LLY US Equity	-0.0091	0.0019	0.0471	2.1767	0.0264	5.0046e-051

JB: Jarque-–Bera test, KS : One-sample Kolmogorov-Smirnov test

Heavy Tailed Distribution The investigation about the nature of the distribution and underlying dynamics of the $log(S_t)$ has been studied in detail in this section. $QQ-$plot 1.4 infers the non-normality because of fat-tails but distribution is still unknown.

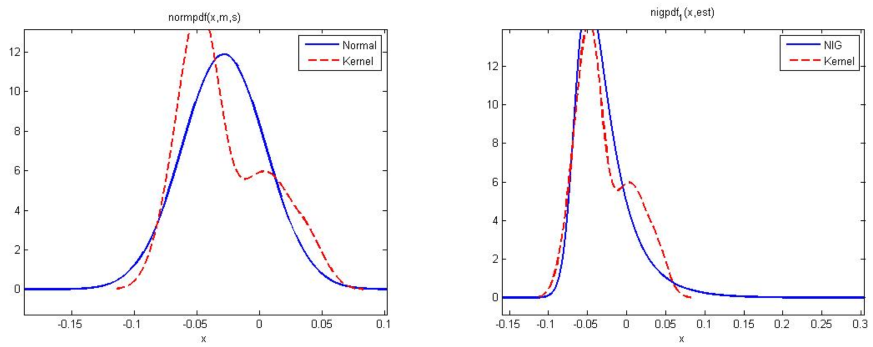

Figure 1.5: Fitting Normal and NIG Distribution of daily stock prices for LLY US Equity

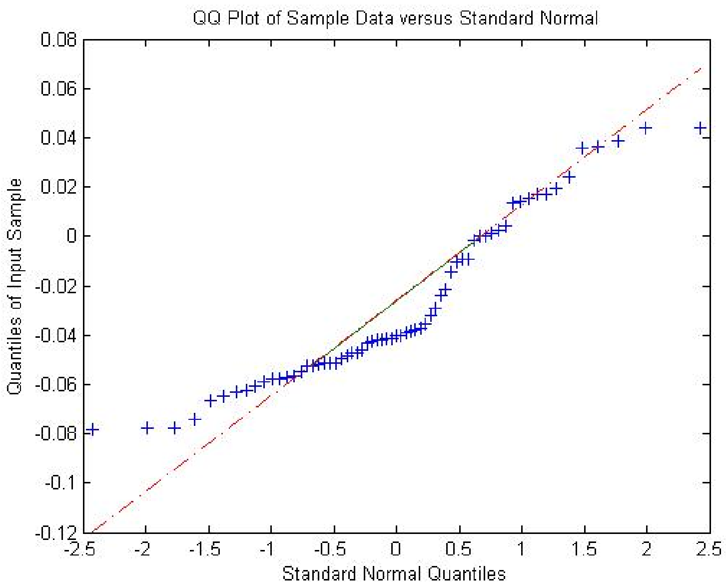

Figure 1.4: Normal QQ-plot of daily stock prices for LLY US Equity.

We study pricing of the option contracts under a known class of Lévy process called NIG[2.3.1] and estimate parameters by Maximum Likelihood method. The distribution shows very close fit in 1.5. The χ^2-test $p - Value$ being close to zero in Table 1.2 confirms the distribution from a class of Lévy process.

Table 1.2: Normal Inverse Gaussian χ^2-test : MLE parameters and P-values

Index	alpha	beta	delta	μ	χ^2 - $P_{NIG} - value$
Google US Equity	-0.1755	0.1125	-0.3282	1.7459	1.3495e-009
MSFT US Equity	-0.0091	0.0439	0.0471	2.1767	5.7176e-004
LLY US Equity	0.0432	0.0564	-0.6161	2.2284	2.6118e-007

Our analysis shows that the general class of Lévy processes are better fit for correct modelling asset price and hence can be well-adopted for pricing and hedging of path-dependent exotic contracts. In the following chapters, we have definitions, stock price modeling and pricing approach for three popular exotic options under Lévy processes with comparative study from literatures.

Chapter 2

Modeling with Lévy process

The following are the basic definitions of Lévy process, It$\hat{o}$-Lévy calculus, arbitrage-free pricing and hedging methods described in the various literatures for modeling stock price and closed-form pricing. Then, we have described the challenges in deriving risk neutral pricing or density for three selected exotic options and our contributions in the same.

2.1 Definitions

Definition 2.1.1. σ-algebra [1] Given $\mathcal{A}$ of subsets of $\mathcal{E}$, there exists a unique σ-algebra denoted $\sigma(\mathcal{A})$ with the following property : if any σ-algebra $\mathcal{F}'$ contains $\mathcal{A}$ then $\sigma(\mathcal{A}) \in \mathcal{F}'$, $\sigma(\mathcal{A})$ is the smallest σ-algebra containing $\mathcal{A}$ and is called the σ-algebra generated by $\mathcal{A}$.

The σ-algebra is a collection of subsets of $E \subset \mathbb{R}^d$ which

1. contains empty set $\emptyset \in \mathcal{E}$,
2. is stable under unions disjoint

$$A_n \in \mathcal{E}; \text{ such that} \bigcup_{n\geq 1} A_n \in \mathcal{E},$$

3. contains complementary of every element: $\forall A,\ A^c \in \mathcal{E}$.

Definition 2.1.2. Borel σ-algebra [1] The σ-algebra generated by all open subsets is called Borel σ-algebra and denoted by set $\mathcal{B}(E)$ or $\mathcal{B}$. An element $B \in \mathcal{B}$ is called Borel set.

Definition 2.1.3. Measure [1] Let $\mathcal{E}$ be a σ-algebra of subset of E. Then $(E, \mathcal{E})$ is called a measurable space. A measure (positive) on $(E, \mathcal{E})$ is defined as a function

$$\begin{aligned} \mu : \mathcal{E} &\quad \to [0, \infty] \\ A &\quad \mapsto \mu(A) \end{aligned}$$

[1]The definition is taken from [27]

such that

1. $\mu(\emptyset) = 0$.

2. For any sequence of disjoint sets $A_n \in \mathcal{E}$.

$$\mu(\bigcup_{n\geq 1} A_n) = \sum_{n\geq 1} \mu(A_n)$$

An element $A \in \mathcal{E}$ is called a measurable set and $\mu(A)$ its measure.

Definition 2.1.4. Probability space and measure [1] A probability space defined by $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is a set of scenarios ω, σ-algebra $\mathcal{F}$ and probability measure $\mathbb{P}$ defined as

$$\mathbb{P} : \mathcal{F} \to [0, 1]$$

$$A \mapsto \mathbb{P}(A)$$

for a measurable set $A \in \mathcal{F}$.

Definition 2.1.5. Absolute continuity [1] A measure μ_2 is said to be absolutely continuous with respect to μ_1 if for any measurable set A

$$\mu_1(A) = 0 \Rightarrow \mu_2(A) = 0.$$

The measures $\mathbb{P}$ and $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ are equivalent or $\mathbb{P} \sim \mathbb{Q}$ if

$$\forall A \in \mathcal{F}, \mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A) = 0.$$

Definition 2.1.6. Radon-Nikodyn theorem [1] If measure μ_2 is absolutely continuous with respect to measure μ_1 then there exists a measurable function $Z : E \to [0, \infty]$ such that for any measurable set A

$$\mu_2(A) = \int_A Z d\mu_1 = \mu_1(Z 1_A).$$

Definition 2.1.7. Equivalence of probability measures [1] If measure μ_2 is absolutely continuous with respect to measure μ_1 and μ_1 is absolutely continuous with respect to μ_2 then μ_1 and μ_2, are said to be equivalent (comparable) measures.

Definition 2.1.8. Random variable[1] A random variable X taking values in $E \subset \mathbb{R}^d$ is a measurable function

$$X : \Omega \mapsto E,$$

where $\omega \in \Omega$ is the scenario of randomness and $X(\omega)$ is the realization in the scenario ω.

2.2 Stochastic Processes

Definition 2.2.1. Stochastic process [1] A stochastic process $(X_t)_{t\in[0,T]}$ is the function of time for each realization of randomness ω such that

$$X(\omega) : t \mapsto X_t(\omega)$$

for each trajectory called sample path of the process. In general it is defined as

$$X : [0,T] \times \Omega \to E.$$

Definition 2.2.2. Cadlag function[1] A Cadlag function is defined as $f : [0,T] \to \mathbb{R}^d$ which is right-continuous with left limits

$$f(t-) = \lim_{s\to t, s<t} f(s) \text{ and } f(t+) = \lim_{s\to t, s>t} f(s)$$

exists and $f(t) = f(t+)$. It can have discontinuity with jumps at time t and can be denoted by $\Delta f(t) = f(t) - f(t-)$.

Definition 2.2.3. Filtration and histories[1] The Filtration or information flow on $(\Omega, \mathcal{F}, \mathbb{R})$ is an increasing family of σ-algebras $(\mathcal{F}_t)_{t\in[0,T]} : \forall t \geq s \geq 0, \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$. $\mathcal{F}_t$ is interpreted as information known at a time t, which increases with time. A probability space $(\Omega, \mathcal{F}, \mathcal{P})$ equipped with a filtration is called a filtered probability space. An $\mathcal{F}_t$-measurable random variable is random variable whose value revealed at a time t.

Definition 2.2.4. Nonanticipating or Adapted process[1] A stochastic process X_t is said to be nonanticipating with respect to the information structure $\mathcal{F}_t$ or $\mathcal{F}_t$-adapted if, for each $t \in [0,T]$, the value of X_t is revealed at time t: the random variable X_t is $\mathcal{F}_t$-measurable.

Definition 2.2.5. Stopping times [1] A nonparticipating random time τ is called stopping time or $\mathcal{F}_t$-stopping time if

$$\forall t \geq 0, \{\tau \leq t\} \in \mathcal{F}_t.$$

Given a stopping time τ and nonanticipating process X_t, a new process $X_{\tau\wedge t}$ which is a process X stopped at τ defined as

$$X_{\tau\wedge t} = X_t \text{ if } t < \tau \; X_{\tau\wedge t} = X_\tau \text{ if } t \geq \tau.$$

Definition 2.2.6. Martingale[1] A cadlag process X_t is said to be a martingale if X is nonanticipating (adopted to $\mathcal{F}_t$), $\mathbb{E}[|X_t|]$ is finite and $\mathbb{E}[X_s|\mathcal{F}_t] = X_t$ for $s > t$.

Definition 2.2.7. Optional processes [1] The optional σ-algebra is the algebra $\mathcal{O}$ generated on $[0,T] \times \Omega$ by all nonparticipating (adapted) cadlag processes. A process $X : [0,T] \times \Omega \mapsto \mathbb{R}^d$ which is measurable with respect to $\mathcal{O}$ is called an optional process.

Definition 2.2.8. Predictable processes [1] The σ-algebra is the algebra $\mathcal{P}$ generated on $[0,T] \times \Omega$ by all nonparticipating (adapted) left-continuous processes. A mapping $X : [0,T] \times \Omega \mapsto \mathbb{R}^d$ which is measurable with respect to $\mathcal{P}$ is called a predictable process.

Definition 2.2.9. Random measures [1] The Poission Process M_t is defined as a counting process : if $T_1, T_2, ...$ is the sequence of jumps times of M, then the number of jumps between 0 and t :

$$M_t = \#\{i \geq 1,\ T_i \in [0,t]\}$$

and the measure N on $[0, \infty]$ of the counting process M_t for any measurable set $A \subset \mathbb{R}^+$:

$$N(\omega, A) = \#\{i \geq 1,\ T_i(\omega) \in A\},$$

where ω is randomness. Then $N(\omega, .)$ is positive real valued measure and $N(A)$ is finite with probability 1 for any bounded set A. The intensity λ of the Poisson process determines the average value of the random measure N : $E[N(A)] = \lambda|A|$ where $|A|$ is the Lebesgue measure of A.

Definition 2.2.10. Compensated random measures [1] The random measure associated with compensated Poisson process $\tilde{M_t} = M_t - \lambda t$ is defined as

$$\tilde{N}(\omega, A) = N(\omega, A) - \int_A \lambda dt = N(\omega, A) - \lambda|A|.$$

$\tilde{N}$ is defined as compensated random measure and the measure $A \mapsto \lambda|A|$ is called compensator of N.

Definition 2.2.11. Jump Measure of Cadlag process[1] The Random measure associated with Cadlag process X_t is called Jump Measure. The countable number of jumps for X_t is defined by $\Delta X_t = X_t - X_{t-} \in A$ where $A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$ and its Jump measure is $N(t, A)$= Number of jumps of occurring between 0 and t whose magnitude belongs to $A = \#\{0 \leq s \leq t, \Delta X_s \in A\}$.

2.3 Lévy Process

Definition 2.3.1. Lévy Process [1] The Lévy process is a Cadlag process X_t on $(\Omega, \mathcal{F}, \mathbb{P})$ and if it has

(i) Independent increments : For every increasing sequence $t_0, .., t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ are independent.

(ii) Stationary increments : $X_{t+h} - X_t$ is independent of t.

(iii) Stochastic Continuity : For $\varepsilon > 0, \lim_{h \to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0$.

Definition 2.3.2. Infinite Divisibility [1] Let X be a real valued random variable with its characteristics function φ_X and a law P_X, hence $\varphi_X = \int_{\mathbb{R}} e^{iux} P_X(dx)$. The law of a random variable X is infinitely divisible if for all $n \in \mathbb{N}$ there exists i.i.d random variable $X_1^{(1/n)}, ..., X_n^{(1/n)}$ such that

$$X = X_1^{(1/n)} + ... + X_n^{(1/n)}.$$

Then the law P_X is

$$P_X = P_X^{(1/n)} * P_X^{(1/n)} * .... * P_X^{(1/n)}$$

is the convolution n times. Then the characteristics function of the infinitely divisible random variable is

$$\varphi_X(u) = \Big(\varphi_{X^{(1/n)}}(u)\Big)^n.$$

Definition 2.3.3. Characteristic function of a Levy process [1] Let X be a Levy process on $\mathbb{R}^d$. There exists a continuous function $\psi : \mathbb{R}^d \to \mathbb{R}$ called the characteristic exponent of X, such that:

$$\mathbb{E}[e^{iz.X_t}] = e^{t\psi(z)}, \quad z \in \mathbb{R}^d.$$

Definition 2.3.4. Jump Measure of Lévy process [1] For every Lévy process X_t one can associate a random measure on $\mathbb{R}^d \times [0, \infty]$ describing jumps of X for any measurable set $B \subset \mathbb{R}^d \times [0, \infty[$, defined as

$$N_X(B) = \#\{t, (X_t - X_{t-}) \in B\}$$

is called the Jump Measure of of Lévy process.

Definition 2.3.5. Lévy Measure [1] Let X_t be a Lévy process on $\mathbb{R}^d$. The Lévy Measure ν on $\mathbb{R}^d$ is the expected number of jumps per unit time and size belongs to A and defined as

$$\nu(A) = E[\#\{t \in [0,1] : \Delta X_t \in A\}], A \in \mathcal{B}(\mathbb{R}).$$

Proposition 2.3.1. ***Itô-Lévy decomposition*** [2] *Let X_t be a Lévy process on $\mathbb{R}^d$ where $b \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and Lévy measure ν on $\mathbb{R}^d \backslash \{0\}$ and satisfying,*

$$\int_{|x|\leq 1} |x|^2 \nu(dx) < \infty \quad and \quad \int_{|x|\geq 1} \nu(dx) < \infty.$$

Then X_t can be written as

$$X_t = bt + \sigma W_t + \int_0^t \int_{|x|\geq 1} xN(ds, dx) + \int_0^t \int_{|x|\leq 1} xN(ds, dx) - t \int_{|x|<1} x\nu(dx)$$

and the triplet (b, σ^2, ν) is called characteristic triplet or Levy triplet of process X_t.

[2] The definition is taken from [75]

Proposition 2.3.2. [1] *Let X_t be a Lévy process with independent increments. Then*

1. $\left(\frac{e^{iuX_t}}{E[e^{iuX_t}]}\right)$ *is a martingale* $\forall u \in \mathbb{R}$.

2. *If for some* $u \in \mathbb{R}$,$E[e^{uX_t}] < \infty$ $\forall t \geq 0$ *then* $\left(\frac{e^{uX_t}}{E[e^{uX_t}]}\right)$ *is a martingale.*

3. *If* $E[X_t] < \infty$ $\forall t \geq 0$ *then* $M_t = X_t - E[X_t]$ *is a martingale.*

4. *If* $Var[X_t] < \infty$ $\forall t \geq 0$ *and let* $M_t = X_t - E[X_t]$, *then* $(M_t)^2 - E[(M_t)^2]$ *is a martingale, where* M *is the martingale defined above.*

Proposition 2.3.3. ***Lévy-Khintchine representation*** [2] *The law P_X of a random variable X is infinitely divisible if and only if there exists a triplet (b, σ^2, ν) where $b \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and Lévy measure ν on $\mathbb{R}^d \backslash \{0\}$ and satisfying*

$$\int_{|x|\leq 1} |x|^2 \nu(dx) < \infty \text{ and } \int_{|x|\geq 1} \nu(dx) < \infty,$$

then

$$E[e^{iuX_t}] = \exp\left[ibu - \frac{1}{2}\sigma^2 u^2 + \int_R (e^{iux} - 1 - iux\mathbb{1}_{|x|<1})\nu(dx)\right].$$

Proposition 2.3.4. ***Lévy process with Finite activity*** [2] *The Lévy process X_t has jumps of finite activity if $\nu(\mathbb{R}) < \infty$.*

Proposition 2.3.5. ***Lévy process with Finite variation*** [2] *The Lévy process X_t has jumps of finite variation if and only if $\int_{|x|\leq 1} |x|\nu(dx) < \infty$. Then X_t can be written as*

$$X_t = bt + \sigma W_t + \int_{\mathbb{R}} xN(t, dx).$$

Proposition 2.3.6. ***Lévy process with Finite moments*** [2] *The Lévy process X_t has jumps of finite first moment if and only if $\int_{|x|\geq 1} |x|\nu(dx) < \infty$. Then it can be written as*

$$X_t = b't + \sigma W_t + \int_{\mathbb{R}} x\tilde{N}(t, dx),$$

where $b' = b + \int_{|x|\geq 1} |x|\nu(dx) < \infty$

Proposition 2.3.7. ***Subordinator*** [2] *The subordinator is an increasing Lévy process with triplet defined as*

$$\nu(-\infty, 0) = 0, \sigma = 0 and \int_{(0,1)} x\nu(dx) < \infty.$$

Proposition 2.3.8. ***Spectrally one-sided*** [2] *The Lévy process X_t is spectrally negative if $\nu(-\infty, 0) = 0$.*

2.3.1 Examples of Lévy process

[3] We have considered the following Lévy processes with infinite activity and

$$\int_{\mathbb{R}} x^2 \nu(dx) \mathbb{1}_{\{x<1\}} < \infty.$$

1. The Normal Inverse Gaussian

   The NIG distribution with parameters $\alpha > 0, -\alpha < \beta < \alpha$ and $\delta > 0, NIG(\alpha, \beta, \delta)$, has a characteristic function

   $$\mathrm{E}\left[e^{iuX}\right] = exp\Big(-\delta\big(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2}\big)\Big).$$

   The Lévy measure is given by

   $$\nu_{NIG}(dx) = \frac{\delta\alpha}{\pi} \frac{\exp(\beta x)\ K_1(\alpha|x|)}{|x|} dx, \tag{2.3.1}$$

   where $K_\lambda(x)$ is the modified Bessel function of third kind with index λ.

   The NIG process has no Brownian component and its Lévy triplet is

   $$\left[\gamma, 0, \nu_{NIG}(dx)\right], \text{ where}$$

   $$\gamma = \frac{2\delta\alpha}{\pi} \int_0^1 \sinh(\beta x)\ K_1(\alpha x) dx.$$

2. The CGMY Process

   The CGMY(C, G, M, Y) distribution is four parameter distribution with characteristic function

   $$\mathrm{E}\big[e^{iuX}\big] = exp\Big(C\Gamma(-Y)\big((M - iu)^Y - M^Y + (G + iu)^Y - G^Y\big)\Big).$$

   The Lévy measure of this process admits the representation

   $$\nu_{CGMY}(dx) \ = \ C\Big(\frac{e^{-Mx}}{x^{1+y}} \mathbb{1}_{x>0} + C\frac{e^{Gx}}{|x|^{1+y}}.\mathbb{1}_{x<0}\Big) dx \text{ when } C, G, M > 0 \text{ and } y < 2.$$

   The CGMY process is a pure jump Lévy process with Lévy triplet

   $$\left[\gamma, 0, \nu_{CGMY}(dx)\right],$$

   where

   $$\gamma \ = \ C\Big(\int_0^1 x^{-Y} e^{-Mx} dx - \int_{-1}^0 |x|^{-Y} e^{Gx} dx\Big).$$

[3] The definition is taken from [84]

3. The Meixner Process

   The Meixner process is defined by $Meixner(\alpha, \beta, \delta), \alpha > 0, -\pi < \beta < \pi, \delta > 0$ then Lévy measure is defined by

$$\nu_{Meixner}(dx) = \delta \frac{\exp(\beta x/\alpha)}{x \ \sinh(\pi x/\alpha)} dx. \tag{2.3.2}$$

   Since $\int_{-1}^{+1} |x|\nu(dx) = \infty$, the process is of infinite variation but moments of all order exists. The first parameter of Lévy triplet

$$\gamma = \alpha\delta \ \tan(\beta/2) - 2\delta \int_1^\infty \frac{\sinh(\beta x/\alpha)}{\sinh(\pi x/\alpha)} dx.$$

   It has no Brownian part and a pure jump part governed by the Lévy measure.

   The Lévy triplet is given by

$$\big[\gamma, 0, \nu_{Meixner}(dx)\big].$$

2.3.2 It$\hat{o}$-Lévy Calculus

2.3.2.1 Stochastic Integral

Theorem 2.3.9. [4] *Let f be a function defined $f : \mathbb{R} \to \mathbb{R}$, then*

1. *The process $\int_0^t \int_{\mathbb{R}\backslash\{0\}} f(x)\tilde{N}(ds, dx)$ is a compensated poisson process with characteristic function*

$$E\Big[\exp\Big(iu \int_0^t \int_{\mathbb{R}\backslash\{0\}} f(x)\tilde{N}(ds, dx)\Big)\Big] = \exp\Big(t \int_{\mathbb{R}\backslash\{0\}} (e^{iuf(x)} - 1)\nu(dx)\Big).$$

2. *If $f \in L^1(\mathbb{R}\backslash\{0\})$, then*

$$E\Big[\int_0^t \int_{\mathbb{R}\backslash\{0\}} f(x)\tilde{N}(ds, dx)\Big] = t \int_{\mathbb{R}\backslash\{0\}} f(x)\nu(dx).$$

3. *If $f \in L^2(\mathbb{R}\backslash\{0\})$, then*

$$Var\Big(\Big|\int_0^t \int_{\mathbb{R}\backslash\{0\}} f(x)\tilde{N}(ds, dx)\Big|\Big) = t \int_{\mathbb{R}\backslash\{0\}} |f(x)|^2\nu(dx).$$

Proposition 2.3.10. ***Martingale-preserving property*** [1] *If S_t is a martingale then for any simple predictable process ϕ_t the stochastic integral $\int_0^t \phi_t dS_t$ is also a martingale.*

[4] The definition is taken from [4]

Definition 2.3.6. Itô-Isometry formula for Brownian Integrals [1]
Let ϕ_t be a simple predictable process satisfying

$$E[\int_0^T |\phi_t|^2 dt] < \infty$$

and W_t be a Wiener process. Then $\int_0^t \phi dW$ is a square integrable martingale and

$$E[\int_0^T \phi_t dW_t] = 0$$

$$E\Big[|\int_0^T \phi_t dW_t|^2\Big] = E\Big[\int_0^T |\phi_t|^2 dt\Big].$$

Definition 2.3.7. Itô-Isometry formula for Compensated Integral[1]
Let ϕ be a simple predictable function $\phi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, then the process $X_t = \int_0^t \int_{\mathbb{R}} \phi(s,y)\tilde{N}(ds,dy)$ defined by compensated integral is a square integrable martingale with $E[\int_0^t X_s ds] = 0$. Then

$$E\Big[|X_t|^2\Big] = E\Big[\int_0^t |\phi(s,y)|^2 \nu(ds,dy)\Big].$$

Definition 2.3.8. Local Martingale [5] Let X be a cadlag adapted process. We say that X is a local martingale if there exists a sequence of stopping times $(\tau_n)_{n\geq 1}$ such that $\lim_{n\to\infty} \tau_n = \infty$ a.s and stopped process X_{τ_n} is a martingale for all n.

Any cadlag martingale is a local martingale but the converse property does not hold true: local martingales are more general than martingales, and the following criteria useful in practice.

Proposition 2.3.11. [5] *Let $M = (M_t)_{t\in\mathbb{T}}$ be a local martingale. Suppose that*

$$E\Big[\sup_{0\leq s\leq t} |M_s|\Big] < \infty, \forall t \in \mathbb{T}.$$

Then M is a martingale.

Proposition 2.3.12. [5] *Let $M = (M_t)_{t\in\mathbb{T}}$ be a local martingale. Then M is a martingale with $E(M_t^2) < \infty$ if and only if $E([M,M]_t) < \infty$.*

Definition 2.3.9. [1] **Semimartingale** A semimartingale is a cadlag local process X having decomposition in the form:

$$X = X_0 + M + A$$

where M is a cadlag local martingale, and A is a adapted process with finite variation.

[5] The definition is taken from [97]

Proposition 2.3.13. [1] *A finite variation process and a (locally) square integrable (local) martingale are a semimartingale.*

Proposition 2.3.14. [1] *All Levy processes are semimartingales.*

All Levy processes are semimartingales because it can be decomposed into a sum of a square integrable local martingale and a finite variation process: this is the Ito-Levy decomposition (Proposition 2.3.1). Every square integrable martingale is a semimartingale. Every processes with finite variations are semimartingales. Semimartingale $M(t)$ is called a local martingale up to time T if there exists an increasing sequence of $\mathcal{F}_t$-stopping times τ_n such that $\lim_{n\to\infty} \tau_n = T$ a.s and $M(t \wedge \tau_n)$ is a martingale with respect to P for all n. The process

$$M(t) = \int_0^t \int_{\mathbb{R}} x\tilde{N}(ds, dx)$$

is a martingale if

$$\int_0^t \int_{\mathbb{R}} x^2 \nu(dx)dt < \infty.$$

2.3.2.2 Functionals of Lévy process

If X is a Levy process then $Y_t = f(t, X_t)$ is not a Levy process anymore but a more generalized process called semimartingale.

Proposition 2.3.15. *Exponential of a Lévy process*[1]
Let X_t be a Lévy process (b, σ^2, ν) such that $\int_{|y|\geq 1} e^y \nu(y) < \infty$. Then $Y_t = \exp X_t$ is a semimartingale (defined in 2.3.9) with decomposition $Y_t = M_t + A_t$ where the martingale part is

$$M_t = 1 + \int_0^t Y_{s-}\sigma dW_s + \int_{[0,t]\times\mathbb{R}} Y_{s-}(e^z - 1)\tilde{N}(ds, dz)$$

and the continuous finite variation drift part is

$$A_t = \int_0^t Y_{s-}\Big[b + \frac{\sigma^2}{2} + \int_{-\infty}^{\infty} (e^z - 1 - z\mathbb{1}_{|z|\leq 1})\nu(z)\Big]ds.$$

and Y_t is a martingale iff

$$b + \frac{\sigma^2}{2} + \int_{-\infty}^{\infty} (e^z - 1 - z\mathbb{1}_{|z|\leq 1})\nu(z) = 0.$$

2.3.2.3 Itô Differentiations

Proposition 2.3.16. ***Itô formula for Lévy process*** [1]
Let X_t be the Lévy process with Lévy triplet (b, σ^2, ν) and $f : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^2$ function. Then

$$f(X_t) = f(X_0) + \int_0^t \frac{\sigma^2}{2} f''(X_s)ds + \int_0^t f'(X_{s-})dX_s$$
$$+ \sum_{0 \le s \le t, \Delta X_s \ne 0} \big[f(X_{s-} + \Delta X_s) - f(X_{s-}) - \Delta X_s f'(X_{s-}) \big].$$

Definition 2.3.10. Quadratic variation[1] The Quadratic variation process of a semimartingale X is the nonanticipating cadlag process defined by

$$[X, X]_t = |X_t|^2 - 2\int_0^t X_{u-}dX_u.$$

Definition 2.3.11. Quadratic covariation[1] Quadratic covariation of two semimartingale X, Y is $[X, Y]_t = X_tY_t - \int_0^t X_{u-}dY_u - \int_0^t Y_{u-}dX_u$.

Definition 2.3.12. Properties of Quadratic variation[1] Quadratic variation has the following properties of semimartingale X

- An increasing process.
- Jumps of $[X, X]$ is $|\Delta X_t|^2$.
- If X is continuous and has finite variation, then $[X, X] = 0$.
- If $X = \sigma W_t$, then $[X, X] = \sigma^2 t$.
- If X is Poisson process, $[X, X]_t = \sum_{0 \le s \le t} |\Delta X_s|^2$.

Definition 2.3.13. Itô formula for Semimartingale[1]
Let X_t be the semimartingale and for any $\mathcal{C}^{1,2}$ function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$,

$$f(X_t) - f(X_0) = \int_0^t \frac{\partial f}{\partial s}(s, X_s)ds + \int_0^t \frac{\partial f}{\partial x}(s, X_{s^-})dX_s + \int_0^t \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, X_{s^-})d[X, X]_s^c$$
$$+ \sum_{0 \le s \le t, \Delta X_s \ne 0} \Big[f(s, X_s) - f(s, X_{s-}) - \Delta X_s \frac{\partial f}{\partial x}(s, X_{s^-}) \Big].$$

Proposition 2.3.17. ***Two-dimensional Itô formula for Semimartingale***[1]

Let X_t, Y_t be the semimartingales and $f \in \mathcal{C}^{2,2}(\mathbb{R}, \mathbb{R})$ function. Then

$$\begin{aligned} f(X_t, Y_t) - f(X_0, Y_0) = &\int_0^t f_x(X_{s-}, Y_{s-}) dX_s + \int_0^t f_y(X_{s-}, Y_{s-}) dY_s \\ &+ \frac{1}{2}\int_0^t f_{x,x}(X_{s-}, Y_{s-}) d[X^c, X^c](s) + \frac{1}{2}\int_0^t d[Y^c, Y^c](s) \\ &+ \int_0^t f_{x,y}(X_{s-}, Y_{s-}) d[X^c, Y^c](s) + \sum_{0 \le s \le t} \big[f(X_s, Y_s) - f(X_{s-}, Y_{s-}) \\ &- f_x(X_{s-}, Y_{s-}) \Delta X_s - f_y(X_{s-}, Y_{s-}) \Delta_s Y \big]. \end{aligned}$$

2.4 Risk-neutral pricing and Hedging

If a portfolio with d assets whose price process $S_t = (S_t^1, S_t^1, ..., S_t^d)$ and amount of assets are $\phi = (\phi^1, \phi^2, ..., \phi^d)$, then value at a time t is $V_t(\phi) = \sum_{k=1}^d \phi^d S_t^k$, capital gain is $G_t(\phi) = \int_0^t \phi_u.dS_u$ and cost of strategy $C_t(\phi) = V_t(\phi) - G_t(\phi) = \phi_t.S_t - \int_0^t \phi_u.dS_u$.

Definition 2.4.1. Self-financing strategy[1] A strategy ϕ is said to be self-financing if cost $C_t(\phi) = 0$ that means :

$$V_t(\phi) = \int_0^t \phi_u dS_u = \phi_0 S_0 + \int_{0+}^t \phi_u.dS_u.$$

Definition 2.4.2. Replicating strategy[1] A replicating (or hedging) strategy for a contingent claim X is a trading strategy ϕ such that $V_T(\phi) = X$. If there exists such as replicating strategy, the contingent claim is said to be attainable (or redundant).

Definition 2.4.3. Admissible strategy[1] An admissible strategy is a self-financing trading strategy ϕ such that $\{V_t^{\mathbb{Q}}(\phi), \mathcal{F}_t, t \in [0, T]\}$ is a martingale under equivalent martingale measure $\mathbb{Q}$.

Proposition 2.4.1. ***Martingale-preserving property***[1] *If S_t is a martingale then for any simple predictable process ϕ the stochastic integral defined by gain process $G_t(\phi) = \int_0^t \phi dS$ is also a martingale.*

Definition 2.4.4. Arbitrage Opportunity[1] An arbitrage opportunity is a self-financing strategy ϕ which can generate positive terminal gain without any probability of intermediate loss :

$$\mathbb{P}(V_t(\phi) \ge 0) = 1. \ \mathbb{P}(V_T(\phi) > V_0(\phi)) \ne 0.$$

Now if $V_0 = 0$,

$$E^{\mathbb{Q}}[V_T(\phi)] = V_0 + E^{\mathbb{Q}}[\int_0^T \phi_t dS_t] = E^{\mathbb{Q}}[\int_0^T \phi_t dS_t].$$

Now, if $E^{\mathbb{Q}}[\int_0^T \phi_t dS_t] = 0$, we can confirm that $E^{\mathbb{Q}}[V_T(\phi)] = 0$ or $\mathbb{Q}[V_T(\phi) \geq 0] \neq 1$. Since $\mathbb{P} \sim \mathbb{Q}$, we can write $\mathbb{P}[\int_0^T \phi_t dS_t \geq 0] \neq 1$ and no arbitrage strategy ϕ exists. Therefore no arbitrage strategy exists if there is an equivalent martingale measure $\mathbb{Q}$ for which discounted stock price is a martingale.

Please note that the law of one price is a rule that the two self-financing strategies with same terminal values must have same values at all other times. The absence of arbitrage (arbitrage-free) is possible only when discounted price of the asset is a martingale under a new measure $\mathbb{Q}$.

Definition 2.4.5. Risk-neutral pricing[1] Any arbitrage-free pricing rule for a contingent claim H can be represented as

$$\Pi(H) = e^{-r(T-t)} E^{\mathbb{Q}}[H|\mathcal{F}_t]$$

where $\mathbb{Q}$ is an equivalent martingale measure (EMM) such that

$$\begin{gathered}\mathbb{Q} \sim \mathbb{P} : \mathbb{Q}(A) = 0 \Leftrightarrow \mathbb{P}(A) = 0, \forall A \in \mathcal{F} \\ \text{and } E^{\mathbb{Q}}[S_T^i|\mathcal{F}_t] = S_t^i, \forall i = 1, ..., d\end{gathered}$$

is called Risk-neutral pricing. The equivalent measure $\mathbb{Q}$ in such case is called risk-neutral measure.

2.4.1 Finding EMM Q

Let $E \in \mathbb{B}(\mathbb{R}^d)$ and $\mathcal{P}_2$ denote the predictable smallest σ-algebra. We define $\mathcal{P}_2(t, E)$ to be the set of all equivalence class of mappings $f : [0, t] \times \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ which coincide almost with respect to $\rho_\Sigma \times \mathbb{P}$, and satisfy the following conditions

1. f is predictable.
2. $\mathbb{P}\left(\int_0^t \int_E |f(s,x)|^2 \rho_\Sigma(ds, dx) < \infty\right) = 1,$

where ρ_Σ is a σ-finite measure on $\mathbb{R}^+ \times E$. Analogously it is possible to define $\mathcal{P}_2(t)$. Let Y be a Lévy type stochastic integral of the form

$$dY(t) = G(t)dt + F(t)dW(t) + \int_{\mathbb{R}\backslash\{0\}} H(t,x)\tilde{N}(dt, dx),$$

where $\sqrt{G(t)}, F(t) \in \mathcal{P}_2(t)$ and $H \in \mathcal{P}_2(t, \mathbb{R}\backslash\{0\})$ for each $t \geq 0$. By Ito formula, we have

$$\begin{aligned} e^{Y(t)} = 1 &+ \int_0^t e^{Y(s-)} F(s) dB(s) + \int_0^t \int_{|x|<1} e^{Y(s-)} (e^{H(,x)} - 1) \tilde{N}(ds, dx) \\ &+ \int_0^t e^{Y(s-)} \Big(G(s) + \frac{1}{2} F(s)^2 + \int_{|x|<1} \big(e^{H(s,x)} - 1 - H(s,x) \big) \nu(dx) \Big). \end{aligned} \tag{2.4.1}$$

$e^{Y(t)}$ is a exponential martingale with $E[e^{Y(t)}] = 1$ if and only if

$$G(s) + \frac{1}{2}F(s)^2 + \int_{|x|<1} \big(e^{H(s,x)} - 1 - H(s,x)\big)\nu(dx) = 0, \tag{2.4.2}$$

almost surely for all $s \geq 0$. Simplyfying, we have

$$e^{Y(t)} = 1 + \int_0^t e^{Y(s-)}F(s)dB(s) + \int_0^t \int_{|x|<1} e^{Y(s-)}(e^{H(,x)} - 1)\tilde{N}(ds, dx). \tag{2.4.3}$$

Now, the goal is to find the equivalent martingale measure $\mathbb{Q}$, on a fixed time interval $[0, T]$, for which $\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{Y(T)}$, for $0 \leq t \leq T$. The following theorem in [75] describes the modified Lévy process under equivalent martingale measure Q.

Theorem 2.4.2. ***EMM $\mathbb{Q}$ by Girsanov's Theorem*** [4] *Let X_t be a Lévy process with triplet (b, σ^2, ν) under $\mathbb{P}$ and its canonical decomposition $X_t = bt + \sigma W_t + \int_0^t \int_{\mathbb{R}} x(N(dt, dx) - \nu(dx)dt)$ and assume that $\mathbb{Q} \sim \mathbb{P}$.*

1. *Then, there exists a deterministic process $F(s)$ and a measurable non-negative deterministic process $H(t, x)$, satisfying*

   $$\int_0^t \int_{\mathbb{R}} |x(e^{H(s,x)} - 1)|\nu(dx)ds < \infty \text{ and } \int_0^t \int_{\mathbb{R}} \sigma^2 F^2(s)ds < \infty$$

   such that the density process η_t has the form

   $$\eta_t = \mathbb{E}\Big[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t\Big] = \exp\Big[\int_0^t F(s)\sigma dW_s - \frac{1}{2}\int_0^t F^2(s)ds \tag{2.4.4}$$

   $$+ \int_0^t \int_{\mathbb{R}} (e^{H(s,x)} - 1)(N(ds, dx) - \nu(dx)ds) - \int_0^t \int_{\mathbb{R}} (e^{H(s,x)} - 1 - H(s,x))N(ds, dx)\Big]. \tag{2.4.5}$$

2. *Conversely, if η_t is a positive martingale of the form above, then it defines a probability measure $\mathbb{Q}$ such that $\mathbb{Q} \sim \mathbb{P}$.*

3. *In both the cases, we have that $\tilde{W}_t = W_t - \int_0^t F(s)ds$ is a $\mathbb{Q}$ Wiener process, $\tilde{\nu}(dx)ds = e^{H(s,x)}\nu(dx)ds$ is the $\mathbb{Q}$-compensator of $N(t, x)$ and X_t has the following canonical decomposition under $\mathbb{Q}$*

   $$X_t = \tilde{b}t + \sigma\tilde{W}_t + \int_0^t \int_{\mathbb{R}} x(N(dt, dx) - \tilde{\nu}(dx)dt) \tag{2.4.6}$$

   where

   $$\tilde{b}t = bt + \int_0^t \sigma^2 F(s)ds + \int_0^t \int_{\mathbb{R}} x(e^{H(s,x)} - 1)\nu(dx)ds. \tag{2.4.7}$$

We know that the associated process e^Y is a martingale and hence $G(t)$ is determined by $F(t)$ and $H(t)$. With respect to the new measure $\mathbb{Q}$, $W_Q(t) = W(t) - \int_0^t F(s)\,ds$ is a Brownian motion and

$$\tilde{N}_Q(t, E) = \tilde{N}(t, E) - \int_0^t \int_E (e^{H(s,x)} - 1)\nu(dx)\,ds,$$

is a martingale (see [4], Section 5.6.3).

Definition 2.4.6. Market completeness [6] A given contingent claim X is said to be *reachable* if there exists a self-financing strategy ϕ such that

$$V_T(\phi) = X$$

with probability 1. In this case we say that the portfolio is a hedging portfolio or replicating portfolio. If all claims can be replicated we say that the market is complete.

According to the pricing rule, the value of the option is defined by replicating self-financing portfolio considering the market as arbitrage free, complete, and perfect hedging is possible. But in real market, the perfect hedging is not possible because of **incompleteness** specially in the presence of jumps and other rare events. The selection of EMM $\mathbb{Q}$ becomes more challenging in such case as there may exist more than one measure $\mathbb{Q}$ violating uniqueness pricing rule. There are various popular approaches for finding the unique risk neutral measure $\mathbb{Q}$ in such an incomplete market.The following are the methods proposed in the literature.

Esscher Transform [31] Let X be a Lévy process with triplet (b, σ^2, ν), assume that Lévy measure ν is such that $\int_{|x|\geq 1} e^{\theta x} < \infty$ for $\theta \in \mathbb{R}$ and $\tilde{\nu}(dx) = e^{\theta x}\nu(dx)$. This transformation is known as Esscher Transform. The Radon-Nikodym derivative corresponding to the change of measure is

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t = \frac{e^{\theta X_t}}{E[e^{\theta X_t}]} = \exp\left(\theta X_t - \ln E[\exp(\theta X_1)]\right). \tag{2.4.8}$$

Minimal entropy martingale measure [31] The relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$ is defined as

$$\mathcal{E}(\mathbb{Q}, \mathbb{P}) = E^{\mathbb{Q}}\Big[\ln \frac{d\mathbb{Q}}{d\mathbb{P}}\Big] = E^{\mathbb{P}}\Big[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}}\Big].$$

This relative entropy is convex function and $\mathcal{E}(\mathbb{Q}, \mathbb{P}) = 0$ if and only if $\frac{d\mathbb{Q}}{d\mathbb{P}} = 1$. The measure $\mathbb{Q}$ is selected by minimizing the relative entropy $\mathcal{E}(\mathbb{Q}|\mathbb{P})$ and finding solution of

$$\inf_{\mathbb{Q}} \mathcal{E}(\mathbb{Q}|\mathbb{P}). \tag{2.4.9}$$

[6] The definition is taken from [98]

Follmer-Schweizer Minimal Measure [31] The approach for selecting $\mathbb{Q}$ is given by Radon-Nikodym derivative defined as

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t = e^{Y(t)}$$

where

$$d(e^{Y(t)}) = e^{Y(t)}P(t)\Big(\sigma dW_t + \int_{\mathbb{R}} x\tilde{N}(dt, dx)\Big) \tag{2.4.10}$$

for some adapted process $P(t)$.

Measure by calibration [1] This approach for finding $\mathbb{Q}$ is based on the calibration from the market prices C_i at $t = 0$ for a set of benchmark options and look for a risk-neutral measure $\mathbb{Q}$ which correctly prices these options:

$$\forall i \in I, C_i = e^{-rT}E^{\mathbb{Q}}[H(S(T_i))].$$

The model calibration is an inverse problem and hence ill-posed. There may be many pricing models that generates same benchmark prices. The popular methods of calibration are (1) Nonlinear least squares and (2) Regularization using relative Entropy. In case (1), calibration is achieved by minimizing quadratic pricing error :

$$\theta^* = \arg\min_{\mathbb{Q}^\theta} \sum_{i=1}^{N} \omega_i |C^\theta(T_i) - C^i|^2,$$

the optimization is done by gradient-based methods numerically. The other approach (2) is by reformulating calibration problem as a nonlinear least squares problem that does resolve the uniqueness and stability issues. This is achieved by adding a penalization term and minimize the functional

$$\mathcal{J}(\theta) = \arg\min_{\mathbb{Q}^\theta} \sum_{i=1}^{N} \omega_i |C^\theta(T_i) - C^i|^2 + \alpha H(\theta) \tag{2.4.11}$$

where $H(\theta) = \mathcal{E}(\mathbb{Q}^\theta|\mathbb{P}^\theta)$, relative entropy of the model.

2.4.2 Hedging Strategy

Hedging is a method of replication of the target payoff of contract with a trading strategy ϕ for minimization of the loss defined by cost process $C(\phi)$. The following are the few definitions and popular hedging methods.

Definition 2.4.7. Perfect Hedging [1] A self-financing strategy (ϕ_t^0, ϕ_t) or replication strategy exists for a contingent claim H such that

$$H = V_0 + \int_0^T \phi_t dS_t + \int_0^T \phi_t^0 dS_t^0.$$

Definition 2.4.8. Complete Market [1] A self-financing strategy (ϕ_t^0, ϕ_t) or replication strategy exists for a contingent claim H such that

$$\mathbb{P}\big[H = V_0 + \int_0^T \phi_t dS_t + \int_0^T \phi_t^0 dS_t^0\big] = 1.$$

Since $\mathbb{P} \sim \mathbb{Q}$, the discounted values of claim is

$$\hat{H} = V_0 + \int_0^T \phi_t d\hat{S}_t$$

and $E^{\mathbb{Q}}[\hat{H}] = V_0$. This is possible when there is only one martingale measure $\mathbb{Q}$. Therefore, complete market must have an unique martingale measure $\mathbb{Q}$.

Definition 2.4.9. Superhedging [1] To replicate a contingent claim H, a conservative approach is to find a self-financing strategy ϕ such that

$$\mathbb{P}(V_T(\phi) = V_0 + \int_0^T \phi dS \geq H) = 1.$$

This is called a superhedging strategy for a claim H with initial capital V_0 and V_T is the value of the portfolio at time T.

In financial theory, a market is considered as complete if option can be replicated by a self-financing strategy with underlying and cash. In such market, options are redundant. Actually, in real markets, perfect hedging is not possible and options allow to hedge risk that cannot be hedged by underlying only. Modeling with Levy or Jump process makes pricing accurate but market becomes incomplete and options are not redundant. In such case, hedging strategy will have residual risk which cannot be hedged away to zero. The following are two methods of hedging incomplete market when underlying S is martingale and semimartingale.

2.4.2.1 Global Risk Minimizing strategy

[1] The goal of Global Risk Minimizing (RM) strategy is to minimize the variance of the cost process C, while the portfolio should equal to contingent claim at payment date. Hence, we minimize at any point of time

$$E[(C_T - C_t)^2 | \mathcal{F}_t],$$

with $C_t = V_t - \int_0^t \xi_u dS_u$ where V denotes the value process of the portfolio and $\int_0^t \xi_u dS_u$ is the gain process of trading in the underlying S.

Consider the risk neutral $\mathbb{Q}$ measure with price dynamics $d\hat{S}_t = \hat{S}_t dZ_t$, where Z_t is a Lévy process and option with payoff $H(S_T)$ is available for risk minimizing hedge. Now, we look for a self-financing strategy for a portfolio (ϕ_t^0, ϕ_t) and initial capital V_0 over a lifetime of a option with contingent claim H which minimizes terminal hedging mean square error such that

$$\inf_{\phi} E[|V_T(\phi) - H|^2] \text{ where } V_T(\phi) = V_0 + \int_0^T r\phi_t^0 dt + \int_0^T \phi_t dt. \tag{2.4.12}$$

Mean-variance Hedging for Exponential-Levy process Let us consider a risk neutral measure $\mathbb{Q}$ and discounted stock price $\hat{S}_t$ is martingale. Consider a self-financing strategy (ϕ_t^0, ϕ_t), then the discounted value of portfolio at terminal

$$
\begin{aligned}
\hat{V}_T &= \int_0^T \phi_t d\hat{S}_t dt = \int_0^T \phi_t \hat{S}_{t-} dZ_t \\
&= \int_0^T \phi_t \hat{S}_{t-} dW_t + \int_0^T \int_{\mathbb{R}} \phi_t \hat{S}_{t-} z \tilde{N}(dt, dz).
\end{aligned}
$$

Let us consider price of the contingent claim H is

$$
C(t, S) = e^{-r(T-t)} E^{\mathbb{Q}}[H(S_T)|\mathcal{F}] = e^{-r(T-t)} E^{\mathbb{Q}}[H(S_T)|S_t = S]
$$

Applying Itô formula to $e^{-rt}C(t, S_t)$, we have

$$
\begin{aligned}
\hat{C}(t, S_t) - \hat{C}(0, S_0) &= \int_0^t \frac{\partial C}{\partial S}(u, S_{u-}) \sigma dW_u \\
&+ \int_0^t \int_{\mathbb{R}} [C(u, S_{u-}(1+z)) - C(u, S_u)] \tilde{N}(dt, dz) \\
&= \int_0^t \frac{\partial C}{\partial S}(u, S_{u-}) \sigma dW_u \\
&+ \int_0^t \int_{\mathbb{R}} [C(u, S_{u-} e^x) - C(u, S_u)] \tilde{N}(dt, dx)
\end{aligned}
$$

where X_t is a Lévy process such that $\hat{S}_t = e^{X_t}$. Therefore, the hedging error

$$
\begin{aligned}
\epsilon(V_0, \phi) &= \hat{C}(T, S_T) - \hat{C}(0, S_0) - \hat{V}_T \\
&= \int_0^T [\phi_t S_{t-} - S_{t-} \frac{\partial C}{\partial S}(t, S_{t-})] \sigma dW_t \\
&+ \int_0^T dt \int_{\mathbb{R}} \tilde{N}(dt, dz)[\phi_t S_{t-} z - (C(t, S_{t-}(1+z)) - C(t, S_t))]
\end{aligned}
$$

Now, the variance of the hedging error $\epsilon(V_0, \phi)$ using Itô-isometry we have

$$
\begin{aligned}
E[\epsilon(\phi)^2] &= E[\int_0^T \int_{\mathbb{R}} \nu(dz) |C(t, S_{t-}(1+z)) - C(t, S_t) - \hat{S}_{t-} \phi_t z|^2] \\
&+ E[\int_0^T dt \hat{S}_{t-}^2 (\phi_t - \frac{\partial C}{\partial S}(u, S_{u-}))^2 \sigma^2]
\end{aligned}
$$

Differentiating the quadratic expression with respect to ϕ_t, we have

$$
\begin{aligned}
&\hat{S_{t-}}^2 \sigma^2 (\phi_t - \frac{\partial C}{\partial S}(t, S_{t-})) + \\
&\int_{\mathbb{R}} \nu(dz) \hat{S}_{t-} z [\hat{S}_{t-} \phi_t z - C(t, S_{t-}(1+z)) - C(t, S_t)|] = 0
\end{aligned}
$$

Mean-variance Hedging for martingale case Let us assume that we have chosen a pricing rule given by risk-neutral measure $\mathbb{Q}$ such that discounted stock price $\hat{S}_t$ is a martingale under $\mathbb{Q}$ and hedging problem simplifies to:

$$\inf_{V_0,\phi} E^{\mathbb{Q}}|\epsilon(V_0,\phi)|^2,$$

where $\epsilon(V_0,\phi) = \hat{H} - V_T = \hat{H} - V_0 - \int_0^T \phi_t d\hat{S}_t$. We assume H has finite variance - $H \in L^2(\Omega, \mathcal{F}, \mathbb{Q})$ and $\hat{S}_t$ is a square-integrable $\mathbb{Q}$-martingale. If we consider portfolios whose terminal values have finite variance:

$$\mathcal{S} = \{\phi \text{ cadlag predictable and } E|\int_0^T \phi_t \hat{S}_t|^2 < \infty\}$$

then set

$$\mathbb{A} = \{V_0 + \int_0^T \phi_t dS_t, V_0 \in \mathbb{R}, \phi \in \mathcal{S}\}$$

of attainable payoffs is a closed subspace of the space of random variables with finite variance, denoted by $L^2(\Omega, \mathcal{F}, \mathbb{Q})$. If there are two random variables X, Y with finite variance are orthogonal such that $E[XY] = 0$, then space becomes a Hilbert space. The mean-variance hedging can be written as

$$\inf_{V_0,\phi} E^{\mathbb{Q}}|\epsilon(V_0,\phi)|^2 = \inf_{A\in\mathbb{A}} ||\hat{H} - A||^2_{L^2(\mathbb{Q})},$$

so the problem of minimizing the mean-square hedging error is equivalent to finding the orthogonal projection in $L^2(\mathbb{Q})$ of the discounted payoff $\hat{H}$ on the set of attainable claims $\mathbb{A}$. This orthogonal decomposition when $\hat{S}_t$ is square-integrable martingale (local martingale) described in Galtchouk-Kunita-Watanabe.

Issue with mean-variance hedging The goal of quadratic hedging is to find risk-neutral measure $\mathbb{Q}$ and hedging strategy ϕ in order to minimize hedging error $E^{\mathbb{Q}}|H - V_T(\phi)|^2$. Unfortunately, $\mathbb{Q}$ represent pricing rule but not the statistical description of market events. Hence the risk neutral variance will be small whereas the portfolio may have large profit and loss (P&L) variance. Therefore, the analysis is done under physical measure $\mathbb{P}$ to find self-financing strategy ϕ and initial capital V_0 to minimize

$$E^{\mathbb{P}}\Big[(V_0 + \int_0^T \phi dS - H)^2\Big].$$

The solution defines linear arbitrage-free pricing rule represented as :

$$V_0 = E^{\mathbb{Q}^{MV}}[\hat{H}]$$

for some martingale measure $\mathbb{Q}^{MV} \sim \mathbb{P}$. Finding such measure is difficult to obtain in case of models with jumps [10].

2.4.2.2 Locally Risk Minimizing strategy

Hedging under restrictions is a problem of great practical importance specially when liquidity is poor and for short-to-delivery contracts which are not traded very far ahead of time. In such case, liquidity constraint either not allow a direct hedging or associated cost is high. A hedging strategy called Locally Risk Minimizing (LRM) outperform than normal(Global) Risk Minimizing (RM) in such cases. Also, when stock price process is a semimartingale, the Global Risk Minimizing strategy is not applicable.

Stock Price as Semimartingale In the finite horizon case, the stock price S is a strict local martingale if there is a bubble [99]. Recently, [100] and [101] have remarked that there is a paucity of examples of strict local martingales with jumps. This assumption is more realistic in practice but our research is restricted to Levy process only, which is a semimartingale.

Local risk minimization for semimartingale When the underlying of the contingent claim that we want to hedge are semimartingales we can no longer apply RM hedging strategy, but need to apply LRM hedging described by Schweizer. Galtchouk-Kunita-Watanabe proposed that the trading strategy ϕ^H is optimal when the contingent claim can be decomposed into a hedgeable part (integral with self-financing strategy) and a residual error which is orthogonal to $\hat{S}$ under $\mathbb{Q} \sim \mathbb{P}$.

Proposition 2.4.3. ***Galtchouk-Kunita-Watanabe decomposition*** [1] *Let $\hat{S}_t$ be a square-integrable martingale with respect to $\mathbb{Q}$. Any random variable $\hat{H}$ with finite variance depending on history $\hat{\mathbb{F}}_t$ of $\hat{S}$ can be represented as sum of a stochastic integral w.r.t $\hat{S}_t$ and a random variable N orthogonal to $\mathbb{A}$: there exists a square-integrable predictable strategy ϕ_t^H such that, with probability* 1

$$\hat{H} = E^{\mathbb{Q}}[\hat{H}] + \int_0^T \phi_t^H d\hat{S}_t + N^H,$$

where N^H is orthogonal to all stochastic integrals w.r.t $\hat{S}$. Moreover, the martingale defined by $N_t^H = E^{\mathbb{Q}}[N^H|\mathcal{F}_t]$ is strongly orthogonal $\mathbb{A}$: for any square-integrable predictable process θ_t, $N_t \int_0^t \theta dS$ is again a martingale.

We have the following facts from the GKW decomposition

- The integral $\int_0^T \phi_t^H d\hat{S}_t$ is the orthogonal projection of random variable H_T on the space of stochastic integrals.

- The random variable N^H represents the residual risk of payoff H that cannot be hedged out. $N^H = 0$ for complete market.

- The random variable $N_t^H = E^{\mathbb{Q}}[N^H|\mathcal{F}_t]$ is orthogonal to self-financing portfolio on $[0, t]$ to optimize global hedging error((2.4.12)) which is a locally risk minimizing strategy.

- Orthogonality is not invarient under change of measure. The choice of martingale measure $\mathbb{Q}$ is adhoc.

Later Follmer and Schweizer proposed a method of decomposition under a unique measure $\mathbb{Q}^{FS}$ and look for locally risk minimizing (LRM) hedging strategy when S is a semimartingale.

Proposition 2.4.4. ***Follmer and Schweizer decomposition*** [1] *If S is a semimartingale under the measure P, then*

$$H = E^{\mathbb{Q}^{FS}}[H] + \int_0^T \phi_t^{FS} dS_t + N^H,$$

is the Follmer and Schweizer decomposition of contingent claim H if the ϕ^{FS} is a square-integrable process, and N^H is square-integrable P-martingale orthogonal to the martingale part of S, with $N_0^H = 0$. For continuous semimartingale the FS decomposition can be deducted from the GKW-decomposition under minimal martingale measure (MMM), which is unique measure such that every local martingale N under P, which is also orthogonal to the martingale part of S remains a martingale under MMM.

2.4.2.3 Hedging by calibration

[1] Consider a contingent claim H and assume that we have as hedging instruments a set of benchmark options with prices $C_i^*, i = 1...n$ and terminal payoffs $H_i, i = 1...n$. A static hedge of H is a portfolio composed from the options $H_i, i = 1...n$. and the numeraire, in order to match as closely as possible the terminal payoff of H :

$$H = V_0 + \sum_{i=1}^{n} x_i H_i + \int_0^T \phi dS + \epsilon,$$

where ϵ is an hedging error representing the nonhedgeable risk. Consider a pricing rule $\mathbb{Q}$, then the claim H is valued under $\mathbb{Q}$ as:

$$e^{-rT} E^{\mathbb{Q}}[H] = v_0 + \sum_{i=1}^{n} x_i e^{-rt} E^{\mathbb{Q}}[H_i].$$

The cost of hedging portfolio is $V_0 + \sum_{i=1}^{n} x_i C_i^*$. So the value of the claim corresponds to the cost of hedging portfolio if the model prices of the benchmark options H_i correspond to their market prices C_i^*:

$$\forall i = 1....n, \;\; e^{-rT} E^{\mathbb{Q}}[H_i] = C_i^*.$$

2.5 Modeling Stock Price

2.5.1 Stock Price under risk neutral measure $\mathbb{Q}$

We denote the stock-price at a given time t by S_t. It is well known that contrary to the assumption of Normality, the log-return of stock-price (that is $\log(S_t)$)) is neither

Gaussian, nor homogeneous and it does *not* have independent increments see e.g. [3]. Our study in 1 also confirms the presence of jumps in the return on the stock price. Therefore, stock price can be represented more accurately by the exponential Lévy process as described below.

$$\begin{aligned} S(t) &= S(0)e^{Z(t)}, \\ dZ(t) &= \mu dt + \sigma dW(t) + \int_{\mathbb{R}} x\tilde{N}(dt, dx), \end{aligned} \tag{2.5.1}$$

with $\tilde{N}(dt, dx) = N(dt, dx) - \nu(dx)dt$, where N is the jump measure of Z and $W(t)$ is the Brownian motion. The Lévy triplet for Z is (μ, σ^2, ν) (with respect to Physical measure $\mathbb{P}$). We also assume that our class of Lévy processes are square-integrable (in the following assumption)

Assumption 2.5.1. $\int_0^t \int_{|x|\leq 1} x^2 \nu(dx)dt < \infty$.

This assumption makes our Levy process also a semimartingale since local martingale is pure martingale under the above assumption 2.5.1. We also make the following assumption related to the nature of the function $H(t, x)$ used while measure transformation in 2.4.1.

Assumption 2.5.2. $\int_{|x|\geq 1} e^x \nu_Q(dx) = \int_{|x|\geq 1} e^{x+H(t,x)} \nu(dx) < \infty$.

Thus with respect to the new measure $\mathbb{Q}$ defined in 2.4.1 the dynamics of Z is given by

$$dZ(t) = \left(\mu + \sigma F(t) + \int_{\mathbb{R}} x(e^{H(s,x)} - 1)\nu(dx)\right) dt + \sigma dW_Q(t) + \int_{\mathbb{R}} x\tilde{N}_Q(dt, dx). \tag{2.5.2}$$

Also,

$$\int_{\mathbb{R}} x\tilde{N}_Q(dt, dx) = \int_{\mathbb{R}} x(N(dt, dx) - \nu_Q(dx)dt),$$

where

$$\nu_Q(dx) = e^{H(t,x)}\nu(dx), \tag{2.5.3}$$

is the Lévy measure with respect to $\mathbb{Q}$. Thus from (2.5.2) and (2.5.3) it is clear that the Lévy triplet of Z with respect to $\mathbb{Q}$ in terms of Lévy triplet with respect to $\mathbb{P}$ is given by

$$\left(\mu_Q, \sigma^2, e^{H(t,x)}\nu(dx)\right), \quad \mu_Q = \mu + \sigma F(t) + \int_{\mathbb{R}} x(e^{H(t,x)} - 1)\nu(dx). \tag{2.5.4}$$

Remark 2.5.3. [2] *In general, Z is not necessary a Levy process under the measure $\mathbb{Q}$; this depends on the tuple (H, F). The following cases exists.*

- *If (H, F) are deterministic and independent of time, then Z remains a Levy process under $\mathbb{Q}$.*

- *If (H, F) are deterministic but depends on time, then Z becomes a process with independent increments under $\mathbb{Q}$. This is a additive process.*

- *If (H, F) are neither deterministic nor independent of time, then Z is a semi-martingale under $\mathbb{Q}$.*

Remark 2.5.4. *Using Girsanov's theorem (see [75]), there exist a deterministic process $\beta(t)$ and a measurable non-negative deterministic process $Y(t, x)$ such that*

$$\mu_Q = \mu + \sigma^2\beta(t) + \int_{\mathbb{R}} x(Y-1)\nu(dx), \quad {\sigma_Q}^2 = \sigma^2, \quad \nu_Q(dx) = Y\nu(dx).$$

Comparing with (2.5.4) *we obtain $\beta(t) = \frac{F(t)}{\sigma}$ and $Y(t, x) = e^{H(t,x)}$.*

With respect to $\mathbb{Q}$, $e^{-rt}S(t) = e^{-rt+Z(t)}$ is a martingale. By Proposition 3.18(2), [27] and (2.5.4), we thus obtain (since we have Assumption 2.5.2)

$$\frac{\sigma^2}{2} + \mu_Q + \int_{\mathbb{R}} (e^x - 1 - x1_{|x|\le 1})\nu_Q(dx) = r. \tag{2.5.5}$$

Therefore the dynamics of stock-price is given by the following theorem.

Theorem 2.5.5. *With respect to the equivalent martingale measure $\mathbb{Q}$, the dynamics of $S(t)$ is given by*

$$\frac{dS(t)}{S(t-)} = r\,dt + \sigma dW_Q(t) + \int_{\mathbb{R}} (e^x - 1)\tilde{N}_Q(dt, dx).$$

Proof. Using results for exponential of a Lévy process (see Proposition 8.20, [27]) we obtain,

$$\frac{dS(t)}{S(t-)} = \left(\mu_Q + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^x - 1 - x1_{|x|\le 1})\nu_Q(dx)\right) dt + \sigma dW_Q(t) + \int_{\mathbb{R}} (e^x - 1)\tilde{N}_Q(dt, dx).$$

Thus the proof follows from (2.5.5). □

2.5.2 Finding $\mathbb{Q}$ ($\mathbb{Q}^{FS}$) using Follmer and Schweizer decomposition

It is clear from (2.5.4) and (2.5.5) that there are non-unique ways (depending on various choices of $F(t)$ and $H(t, x)$) of selecting density function Y. The choice for the equivalent martingale measure $\mathbb{Q}$ in this will be the *Föllmer-Schweizer minimal* *measure*(2.4.4). In this procedure there is an unique measure $\mathbb{Q}$ [31] that minimizes the hedging error $E[|V_t(\phi) - H|^2]$ locally such that $\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} = e^{Y(t)}$, where

$$d(e^{Y(t)}) = e^{Y(t)}\ P(t) \left(\sigma dW(t) + \int_{\mathbb{R}} x\tilde{N}(dt, dx) \right), \tag{2.5.6}$$

for some adapted process $P(t)$. Comparing 2.5.6 and 2.4.3, we have

$$\sigma P(t) = F(t), \quad xP(t) = e^{H(t,x)} - 1,$$

for $t \geq 0$ and $x \in \mathbb{R}$. We define

$$\rho_1 = \int_{\mathbb{R}} x^2 \nu(dx), \quad \rho_2 = \int_{\mathbb{R}} x(e^x - 1 - x1_{|x|<1})\nu(dx), \quad \rho_3 = \int_{\mathbb{R}} (e^x - 1 - x1_{|x|<1})\nu(dx).$$

Then we obtain the following expression from (2.5.5).

$$P(t) = \frac{r - \mu - \frac{\sigma^2}{2}}{1 + \rho_1 + \rho_2 + \rho_3} = \rho. \tag{2.5.7}$$

We note that given r and the Lévy triplet of Z with respect to measure $\mathbb{P}$, i.e., (μ, σ^2, ν), (2.5.7) gives a *constant function*, for $P(t) = \rho$. Thus, $F(t) = \sigma\rho$ is also constant. On the other hand, $H(t, x)$ is a function of x alone and it is given by $H(t, x) = \log(1 + \rho x)$. Consequently, the Lévy density $\nu_Q(dx) = (1 + \rho x)\nu(dx)$. The derived parameter ρ is also known as the *market price of risk* for the Lévy market. The pair of unknown function (H, F) is deterministic and independent of time so Z remains a Levy process under new measure $\mathbb{Q}$.

For the convenience of notation, in this section, we write simply W and $\tilde{N}$ in lieu of W_Q and $\tilde{N}_Q$ respectively. Since in this section we mostly work with the equivalent martingale measure $\mathbb{Q}$ this abuse of notation will not create any confusion. However, we will keep the notation for the Lévy density with respect to $\mathbb{P}$ and $\mathbb{Q}$ as the same as in the previous section, viz. ν and ν_Q respectively. For the Föllmer Schweizer minimal equivalent martingale measure $\mathbb{Q}$,

$$\nu_Q(dx) = (1 + \rho x)\nu(dx),$$

where ρ is given by (2.5.7). Also, assume the Lévy density corresponding to Lévy measures ν_Q and ν are denoted as $w_Q(x)$ and $w(x)$ respectively. Thus for the Föllmer Schweizer case

$$w_Q(x) = (1 + \rho x)w(x). \tag{2.5.8}$$

2.6 Problem of pricing Exotic options

There is a huge interest in research community to develop an arbitrage-free pricing method for the most popular path dependent exotic options such as Asian, Barrier and Look-back with the dynamics defined in (2.5.1) under a risk neutral measure $\mathbb{Q}$. The pricing definition of these contracts are discussed in the following paragraphs.

Asian Option Let $A(t) = \int_0^t S_u \, du$. Then A is an increasing continuous process and thus has no Brownian component. There are four different types of arithmetic Asian options according to the payoff function. let T be the time of expiry of the option. For *fixed strike* (E) call and put Asian options payoffs are given by $\left(\frac{1}{T}A(T) - E\right)^+$ and $\left(E - \frac{1}{T}A(T)\right)^+$ respectively. For *floating strike* call and put Asian options the payoffs are given by $\left(S(T) - \frac{1}{T}A(T)\right)^+$ and $\left(\frac{1}{T}A(T) - S(T)\right)^+$ respectively. Under an equivalent martingale measure $\mathbb{Q}$, the put price of floating Asian option type can be written as

$$P(t, S(t), A(t)) = e^{-r(T-t)}\mathbb{E}^Q\Big[\Big(\frac{A(T)}{T} - S(T)\Big)^+ |\mathcal{F}_t\Big].$$

Up-And-Out and Down-And-Out Barrier Option Let S be the stock price and B is a fixed single Barrier. In general, there are four different types of Barrier options according to the payoff functions. Let T be the time of expiry of the option. For *fixed strike* (K) call and put Up-And-Out Barrier options payoffs are given by $(S-K)^+, 0 \leq S \leq B$ and $(K-S)^+, 0 \leq S \leq B$ respectively. For *fixed strike* call and put Down-And-Out Barrier options the payoffs are given by $(S-K)^+, B \leq S$ and $(K-S)^+, B \leq S$ respectively. Under an equivalent martingale measure $\mathbb{Q}$, the Up-And-Out and Down-And-Out Barrier call option can be written as

$$\begin{aligned} C(t, S(t)) &= e^{-r(T-t)}\mathbb{E}^{\mathcal{Q}}\Big[H(S_T)|\mathcal{F}_t\Big] \text{ where} \\ H(S(T)) &= \Big(S(T) - K\Big)^+ \mathbb{1}_{S(T)\leq B} \text{ for Up-And-Out option} \\ &= \Big(S(T) - K\Big)^+ \mathbb{1}_{S(T)\geq B} \text{ for Down-And-Out option} \end{aligned}$$

Look-back option Let maximum process $M(t) = \max_{0\leq u\leq t} S(u), \quad 0 \leq u \leq t$ is strictly increasing process and also has no Brownian component. In general, there are four different types of look-back options according to the payoff function. Let T be the time of expiry of the option. For *fixed strike* (E) call and put look-back options payoffs are given by $(M(T) - E)^+$ and $(E - M(T))^+$ respectively. For *floating strike* call and put look-back options the payoffs are given by $(S(T) - M(T))^+$ and $(M(T) - S(T))^+$ respectively. Under an equivalent martingale measure $\mathbb{Q}$, the floating type Look-back put option can be written as

$$P(t, S(t), M(t)) = e^{-r(T-t)}\mathbb{E}^{\mathcal{Q}}\Big[\big(M(T) - S(T)\big)^+ |\mathcal{F}_t\Big]$$

where $M(T)$ is given by $M(T) = \max_{0\leq u\leq T} S(u), \quad 0 \leq u \leq T$.

2.7 Previous methods and our approach

There are two major difficulties in arbitrage-free pricing of exotic options with Lévy process.

1) Firstly, the derivation of the closed-form pricing expression because the distribution of the payoff function is unknown when stock price follows exponential Lévy process.

2) Secondly, the estimation of risk neutral density Q from the market prices through calibration methods. In order to address these challenges, we have proposed a novel approach for finding an arbitrage-free pricing expression (closed-form) under Q and also a method based on optimal control theory to estimate risk neutral density from the market prices available.

2.7.1 Arbitrage-free closed-form Pricing Expressions

There is a huge interest in the research community for pricing exotic options such as Asian, Barrier and Look-back options under Lévy process and lot of works have been done in the last decade. The most common approach for pricing exotic options is by numerical methods. The main reason for adopting numerical methods is that the closed-form expression is very hard to derive for options with nonlinear payoff functions under a generic class of Lévy process, specially when the distribution of payoff function is unknown. One of the popular method is Monte-Carlo simulation which simulates a huge number of paths of the underlying Lévy process and the stock price. For every path the payoff function is calculated with discounts, and finally, averages over all the paths to obtain a Monte-Carlo estimate of the value of the option. The generation of the paths of the driving Lévy process is described in [82]. The simulation schemes for the VG and NIG setting based on the construction of Gamma and Inverse Gaussian bridges respectively, can be found in [[77],[78]] (see also [59]). A Compound-Poisson approximation in order to simulate a general Lévy process is also described in [1]. Unfortunately, the simulation methods are difficult and computationally expensive in many cases, and also suffers from error convergence issues.

Many authors also proposed other numerical techniques such as numerical inverse of Integral transforms like Fourier, Laplace, Mellin etc for pricing the complex exotic options. Few authors proposed method of moments to find out the distribution of the underlying stock dynamics for pricing and others took the path of discretization of relevant Partially Integro Differential equation for computing the prices. [9] and [23] proposed methods through characteristic function to find risk neutral density for pricing. [79] proposed a method using Fast Fourier Transforms which can compute option surfaces very fast and later bilateral Laplace transforms were used for the valuation of plain vanilla European call and put option. A similar method were also proposed in [66] and extensions, unification and error bounds discussed in [67]. In the following sections, we will discuss about previous approaches in detail along-with our proposed method for pricing and calculating sensitivities of most popular exotic options namely Asian, Barrier and Look-back.

Pricing expression for Asian Option of Arithmetic Type [91] first developed a partial integro-differential equation (PIDE) approach for pricing Asian option of arithmetic type with exponential Lévy models. [2] proposes a numerical method to determine moments of the arithmetic sum of stock price and then replace it by a more tractable distribution with identical first moment assuming the Lévy market as risk neutral and complete. The method works for Lévy processes having finite moments only. [95, 96] propose another numerical approach based on Fourier cosine expansions to compute the price with numerical accuracy.

We proposed a novel method [[28]] where a partial integro-differential equation (PIDE) was derived for pricing using Martingale method and then we applied Fourier transform to get a closed-form expression for floating Asian options price and sensitivities. The pricing method is calibrated with the market prices of the contracts and the accuracy is measured. The procedure is less involved and easy to calculate for all class of Lévy processes. The approach is described in detail in chapter 3. A quick comparison of the previous works with our novel approach is in Table (2.1).

Table 2.1: Asian Option of Arithmetic Type Under Lévy Process

Publication	Method	Challenges
Vecer and Xu (2004) [91]	Semimartingale Model	No closed-form
Albrecher (2004) [2]	Moment-matching, Edgeworth series, static superhedging	No closed-form, Computationally expensive
Fusai and Attilio (2008) [43]	Moments and recursive density function	No closed-form, Computationally expensive
Zhang and Oosterlee(2013) [96]	Characteristic and recursive Fourier cosine transform	No closed-form, Computationally expensive
New Method [28]	**PIDE and closed-form Fourier expression**	

Pricing expression for Barrier Option The first-passage time distributions and Barrier option prices in Lévy models have been investigated in number of papers. [56] proposed a Laplace transformed based approach to compute the prices and greeks of Barrier options for a class of Lévy process with Wiener-Hopf factorisation. For spectrally one-sided Lévy processes with a Gaussian component [80] derived a method to evaluate first-passage distributions. [61], [68] and [85] followed a transform approach to obtain Barrier option prices for jump-diffusion with exponential jumps. [24] investigated discretisation of the associated integro-differential equations in the setting of infinite activity Lévy processes with jumps in two directions. [14] employed Fourier methods to investigate Barrier option prices for Lévy processes of regular exponential type. These approaches are quite involved and based on extremely complex techniques and applicable for specific class of Lévy process.

We have proposed a novel approach [[29]] where a partial integro-differential equation (PIDE) for pricing using Martingale method is derived and then applied Mellin transform to get a closed-form expression for option and its sensitivities. This new method is calibrated with the market prices of the contracts and the model accuracy is measured. The approach is described in detail in chapter 4. The method is simple, easy to calculate and takes less computational time for all class of Lévy process. A quick comparison of the previous approaches with our novel method is in Table (2.2).

Table 2.2: Barrier Option Under Lévy Process

Publication	Method	Challenges
Schoutens (2003) [82]	Monte Carlo Simulation	Computational Time
Boyarchenko and Levendorskii(2002)[14]	The Wiener–Hopf factorization and Pricing Expression	Difficulty in Factorization
Jeannin and Pistorius (2010)[56]	The Wiener–Hopf factorization, First Passage and Laplace transform	Difficulty in Factorization
Kudryavtsev and Levendorskii(2011)[65]	Wiener-Hopf factorization	Difficulty in Factorization
New Method[29]	**PIDE and closed-form Mellin expression**	

Pricing expression for Look-back Option The first proposed Laplace transform based approach on pricing path dependent option like Barrier and Look-back for Brownian model and jump-diffusion models was by [74]. [73] derived formulas in terms of the Wiener-Hopf factors for the Laplace transform of continuously monitored barrier and Look-back options. Later, the probabilistic approach used to recover the results for barrier options derived in [15] using the analytical form of the Wiener-Hopf factorization method. [62] also suggested method by simulating the joint law of the maximum using Wiener–Hopf Monte Carlo technique to calculate the Look-back options. [49] presented a method for calculating Look-back options in case of discretely monitored price using Wiener–Hopf Technique.

The jump diffusions methods such as exponentially distributed Poisson jumps (a double-exponential jump diffusion process (DEJD) and its generalization a hyper-exponential jump-diffusion model (HEJD)) used for pricing with the help of Laplace transform method. The pricing expressions for DEJD model were given by [69] and [63], and for double-barrier options by [86] for HEJD, see [[25], [26], [56]].

Using the Hilbert transform, [[44], [45]] proposed a new computationally efficient method for pricing discrete barrier and Look-back options and calculation of exponential moments of the discrete maximum of a Lévy process. [64] developed a fast and accurate numerical method labelled Fast Wiener-Hopf factorization method (FWHF-method) for pricing continuously monitored barrier options under Lévy processes of a wide class. [11] suggested an enhanced numerical realization of the FFT, which improves the convergence of the FWHF-method. [12] and later [13] developed fast and accurate techniques for calculating prices of finite lived double barrier options under regime-switching HEJD models, and double-barrier options under a general Lévy process.

In our proposed approach [[30]], we have derived a partial integro-differential equation (PIDE) for pricing using Martingale method and then applied Fourier transform to get a closed-form expression for pricing and sensitivities of floating Look-back options. The model is then validated against monte carlo simulation results and accuracy is measured with change of different parameters. We have also proposed a method based on optimal control theory for estimation of risk neutral measure Q required for arbitrage-free by calibration with the observed market prices(in our case simulated price). The pricing approach is simple, less involved and easy to calculate with less computational time and also works for all class of Lévy process. This is described in detail in chapter 5. A quick comparison of the previous approaches with our novel method is in Table (2.3).

Table 2.3: Lookback Option Under Lévy Process

Publication	Method	Challenges
Kudryavtsev and Levendorskii(2011)[65]	Wiener-Hopf factorization	Difficulty in WH-factorization
Haslip and Kaishev (2014)[54]	Fourier Transform B-spline	Difficulty in computing characteristic function with Spitzer-recurrence expansion
New Method[30]	**PIDE and closed-form Fourier expression**	,

2.7.2 Estimation of Risk Neutral Measure Q by Calibration

The estimation of risk neutral density Q defined in [subsection 2.4.1] is one of the main research areas for arbitrage-free pricing. Our study regarding the distribution of stock price in [section 1] established the fact that it has Heavy-tailed feature and follows Lévy process. Unfortunately, the modeling with the Lévy type jump process lead to different challenges. [39] showed that the introduction of jumps in the pricing method leads to a problem of incomplete market and contingent claim cannot be perfectly hedged or replicated. In such case, market may have multiple martingale measures and selection of an unique equivalent (risk-neutral) martingale measure Q becomes an important issue. There are well-known approaches for finding Q such as Esscher transform, Relative Entropy and Calibration described in [27][Chapter 9.5, Chapter 9.6 and Chapter 13]. [46]] also proposed a method of finding Minimal Martingale Measure Q that minimizes local risk of replicating portfolio. It is believed that the measure Q is determined by the market only and it should be estimated directly from the market prices. Therefore, we proposed a new calibration method based on optimal control theory and it is described in [section 5.4].

Chapter 3

Asian Options of Arithmetic type

3.1 Introduction

Asian options are securities with payoffs that depend on the average value of an underlying stock-price over its life-span. These options are fully path-dependent exotic options that have payoffs which depend on the history of the random walk of the asset price via some sort of averages. These options were first successfully priced in 1987 by David Spaughton and Mark Standish of Bankers Trust. Their payoff is typically based on arithmetic or geometric mean of underlying asset prices at monitoring dates before maturity. Pricing Asian options of arithmetic type is difficult even for the simplest asset price model, as the arithmetic average of a set of lognormal random variables is not log-normally distributed. For Asian options payoff depends on the average value of the underlying asset and hence, volatility in the average value tends to be smoother and lower than that of the plain vanilla options. The average is less exposed to sudden crashes or rallies in stock price and over time is harder to manipulate than a single stock price. Thus, the Asian options are less expensive than comparable plain vanilla options.

However, these options have proved to be much more difficult to price than other options. As a result, there are many techniques developed in the literature to address the problem. Geman and Yor, in [50], obtain an analytical formula for pricing Asian options using the Laplace transform. However, this transform exists only for some particular cases. Rogers and Shi in their paper [81] reduce the PDE for Asian options to the PDE in two variables, and use numerical procedure to solve it. Also, they derive lower-bound formulas for Asian options by computing the expectation based on some zero-mean Gaussian variable. Zhang [103] presents a theory of continuously-sampled Asian option pricing and he solves the PDE with perturbation approach. Yang et al. [94] derive quasi analytical expressions for price and hedge arithmetic Asian call option, and Elshegmani et al. [42] derive a modified arithmetic Asian option PDE, together with its analytical solution. There are several approaches to this problem. One can use Monte Carlo simulation techniques to obtain numerical estimates of the price, which can be achieved by adapting procedures developed for the

Black-Scholes case (see [17, 18, 83]). The Fast Fourier Transform approach (for simple Brownian case) is presented in [19]. Another alternative is to use approximations of the distribution of the average, which sometimes leads to closed-form expressions for the price approximation. Different authors (see [70, 67, 89, 92, 87]) discuss adaptions of such approximation technique developed for the Black-Scholes case. All of the above approaches by different authors based on the Black-Scholes framework.

Vecer and Xu [91] first developed a partial integro-differential equation (PIDE) approach that is applicable for exponential Lévy models. Albrecher [2] proposes a numerical method to determine moments of the arithmetic sum of stock price and then replace it by a more tractable distribution with identical first moment assuming the Lévy market as risk neutral and complete. The method works for Lévy processes having finite moments but it is not clear in the case of infinite activity Lévy models. Also, no work is done in that case on estimation of *market price of risk* and model calibration. Zhang and Oosterlee [95, 96] propose a pure numerical approach based on Fourier cosine expansions to compute the price with numerical accuracy and computational time. The methods do not consider the incomplete market with discovery of market price of risk from market value of option and there is no near closed form expression for the Fourier transform which can be used easily to construct prices. The approach is purely numerical and will be extremely difficult to implement for all class of Lévy processes.

Explicit closed-form valuation of the Asian options with the Lévy process is not possible and is restricted to numerical procedures such as PIDE discretization or method of moments. The method is to discretize the PIDE in asset-time domain and use binomial trees, finite difference or finites element methods to solve it. There are examples similar to [65], where an explicit multinomial tree based approach is used for the option valuation. Hilliard and Schwartz [53] develop another pricing procedure based on a lattice framework. Further recent developments concerning the numerical evaluation proposed in the articles such as [32, 23, 33, 41, 72]. The finite difference method for partial integro-differential equations [5] takes $\mathcal{O}(n \log n)$ (n is number of discretization based on value of parameter range, which may be large for a wide range of parameters) computations for each iteration and may take large number of iterations to converge due to non-locality, resulting high computational time. Therefore, the Fast Fourier Transform (FFT) is one of the better numerical approaches adapted if there is any closed form expression of Fourier transforms available for all class of Lévy processes. Unfortunately, we do not have such expressions and the earlier authors take the path of pure numerical techniques, which involve high amount of computations, to compute the option prices. But, we are able to come up with PIDE and a closed form expression of Fourier transform so that the pricing becomes much more easy and elegant. Also as main part of the approach is stochastic, we are able skip major numerical calculations in the approach.

Summarizing all the issues in the previous work, we find few challenges in pricing the Asian option of arithmetic type under Lévy processes. First of all, the Lévy market is incomplete and there are more than one measure exists leading to multiple prices for a single contract and hedging is not possible. Therefore, the pricing model demands selection of the measure and finding market price of risk with the help of market price available by calibration method with better goodness of fit. Secondly, the main difficulty in calculating the value of the Asian option of arithmetic type is to determine the distribution of the arithmetic average of stock prices, for which in general no explicit analytical expression is available. Finally, it is difficult to derive a closed form expression of the Fourier transform of contract so that we can apply well accepted inverse FFT to calculate the price with less computational time, efficiency and accuracy. A model is proposed and a pricing approach is derived based on PIDE, the Fourier transform and its inverse. The advantage of our model is that it has a closed form expression of the Fourier transform applicable for any class of Lévy processes and the standard inverse Fourier transform can be applied to construct prices.

The Fourier transform based method for option pricing was first proposed in papers such as [47, 76]. There are a couple of advantages of the Fourier Transform over other transforms such as the Laplace transform. First, there are properties which can translate a PIDE into algebraic function which can be used for inversion later. Secondly, the FFT and the inverse fast Fourier transform (IFFT) are much faster and do not depend on the range of discretization of the parameters used.

In this paper, we first develop an option pricing PIDE for Lévy processes and apply the Fourier transform to obtain a pricing formula. After that we discuss the calibration method using available market data. At the end, we present the numerical results with computational time and other performance measures.

The organization of different sections in this paper is as follows. Section 2 recalls some basic facts about exponential Lévy processes and provides a model used in this paper. Section 3.3 derives the partial integro-differential equation (PIDE) for the option pricing of floating type Asian put options. It also provides a formula for pricing and sensitivities in terms of the inverse Fourier transform. Numerical results are provided in Section 3.4.

3.2 Benchmark Models

3.2.1 Asian Options in a Semimartingale Model

The proposed method described in [91] is based on the Integro-differential equation when the underlying stock is driven by special semimartingale processes with independent increments and the Lévy process being a special case. The pay-off function for option is

$$\xi(\int_0^T S_t d\lambda - K_1 S_T - K_2)^+ \tag{3.2.1}$$

where $\lambda(t) = \frac{t}{T}$ when continuous and $\lambda(t) = \frac{1}{n}\lfloor \frac{nt}{T} \rfloor$, $\xi = \pm 1$ for call or put, $K_1 = 0$ for fixed and $K_2 = 0$ for floating type. The approach first proposed is a self-financing replication strategy of the Asian forward contract and is given by the following proposition.

Proposition 3.2.1. *If* $q_t = e^{-rT}\int_t^T e^{rs}d\lambda(s)$ *shares invested in the stock with initial value of portfolio*

$$X_0 = q_0 S_0 - e^{-rT}K_2$$

and self-financing portfolio defined by

$$dX_t = q_t dS_t + r(X_{t_-} - q_{t_-}dS_{t_-})dt,$$

then final wealth is

$$X_T = \int_0^T S_t d\lambda(t) - K_2.$$

For pricing Asian options, we can apply the change of numeraire technique introduced in [102]. Let us consider a new measure $\mathbb{Q}$ defined as $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{S_t}{S_0 e^{rt}}$ and the numeraire process $Z_t = \frac{X_t}{S_t}$ is a local martingale under $\mathbb{Q}$. Now, if we model stock price as

$$dS_t = S_{t-}dH_t \tag{3.2.2}$$

where H_t is a semimartingale and its canonical decomposition defined as

$$H_t = rt + H_t^c + \int_0^t \int_{-\infty}^{\infty} x(N(ds,dx) - \nu(ds,dx)) \tag{3.2.3}$$

where H_t^c is the continuous martingale part, $N(ds,dx)$ is the random measure associated with the jumps of H and $\nu(dt,dx)$ is the compensator. Then Doleans-Dade formula gives

$$S_t = S_0\mathcal{E}(H) = S_0 \exp(H_t - \frac{1}{2}\langle H_t^c\rangle) \prod_{0<s\le t}(1+\Delta H_s)e^{-\Delta H_s}$$

The pricing methods of Asian options is given by the following Integro-differential equation

Theorem 3.2.2. *The value of the Asian option is* $V^\lambda(0,S_0,K_1,K_2) = S_0 v(0,Z_0)$ *where for function* $v(t,Z_t)$*, the derivatives* v_t, v_z, v_{zz} *are continuous and defined by the following partial differential equation*

$$\int_0^t \Big[v_s(s,Z_{s_-}) + \frac{1}{2}v_{zz}(s,Z_{s_-})(q_- - Z_{s_-})^2 d\langle H^c\rangle_s \tag{3.2.4}$$

$$+\int_{-\infty}^{\infty}\Big\{v(s,Z_{s_-} + (q_- - Z_{s_-})\frac{x}{1+x}) - v(s,Z_{s_-}) - v_z(s,Z_{s_-})(q_- - Z_{s_-})\frac{x}{1+x}\Big\}\nu(ds,dx)\Big] = 0 \tag{3.2.5}$$

for $0 \le t \le T$

3.2.2 Asian Options Under Exponential Lévy type

The price of a European-style arithmetic average call option at time t under exponential Lévy models given by

$$AA_t = \frac{e^{-r(T-t)}}{n}\mathbb{E}^{\mathbb{Q}}\Big[\Big(\sum_{k=0}^{n-1} S_{T-k} - nK\Big)^+ |\mathbb{F}_t\Big] \tag{3.2.6}$$

where where n is the number of averaging days, K is the strike price, T is the time to expiration, $\mathbb{F}_t$ is the information available at time t and $\mathbb{Q}$ any risk-neutral pricing measure.
The proposed method described in [2] calculate the moments of $A_n = \sum S_k$ by following recursive ways and finds the distribution by moment-matching technique with the help of Edgeworth series expansion. The sum A_n can be defined as

$$\sum S_k = S_0(R_1 + R_1R_2 + ... + R_1R_2..R_n) = S_0R_1L_1 \tag{3.2.7}$$

where $R_i = \frac{S_i}{S_{i-1}}, i = 1, ..., n$ and $L_{i-1} = 1 + R_iL_i, i = 2, ..., n$ with $L_n = 1$. Then the moments of A_n is $\mathbb{E}^{\mathbb{Q}}[(A_n)^m] = S_0^m\mathbb{E}^{\mathbb{Q}}[R_1^m]\mathbb{E}^{\mathbb{Q}}[L_1^m]$ and $\mathbb{E}^{\mathbb{Q}}[L_1^m]$ can be calculated from the following recursive expression

$$\mathbb{E}^{\mathbb{Q}}[L_{i-1}^m] = \mathbb{E}^{\mathbb{Q}}[(1 + L_iR_i)^m] = \sum_{k=0}^{m}\binom{m}{k}\mathbb{E}^{\mathbb{Q}}[L_i^k]\mathbb{E}^{\mathbb{Q}}[R_i^k] \tag{3.2.8}$$

where $\mathbb{E}^{\mathbb{Q}}[L_n^k] = 1, k = 0, ..., m$. These moments are used for finding the distribution of sum A_n by Edgeworth series expansion and option value computed numerically.
The method proposed a static super-hedging for fixed-strike Asian call option based on buy-hold strategy. They have shown that the upper and lower analytic bounds for the Asian call option price can be represented by price of a European call option at time different with strike and maturity as follows

$$AA_0 \leq \frac{1}{n}\sum_{k=1}^{m} \exp\left(r(T - t_k)\right)EC_0(\kappa_k, t_k) \tag{3.2.9}$$

where $K = \sum_{k=1}^{n} \kappa_k$ and $EC_0(n\kappa_k, t_k)$ is the price of the a European call option at time 0 with strike $n\kappa_k$ and maturity t_k. Therefore buying $\exp(-r(T - t_k))EC_0(\kappa_k, t_k)$ European call options at time $t = 0$ (with strike κ_k, maturity t_k) $(k = 1, ..., n)$, holding them until their expiry and putting their payoff on the bank account represents a static superhedging strategy for this Asian option.

3.2.3 Pricing Discretely Monitored Asian Options under Lévy Processes

The proposed approach described in [43] for pricing arithmetic Asian options valuation is based on a recursive integration method and recursive formula the moments. The pay-off of an arithmetic Asian option depends on the following path-dependent random variable

$$A_T^\Delta = \frac{1}{T+1}\sum_{k=0}^{T} S_{\Delta k}$$

The pay-off of an arithmetic Asian call option is

$$C_f x^a(K,T) = max\{A_T^\Delta - K, 0\}.$$

Let us consider

$$S_{T\Delta} = S_0 e^{mT\Delta + X_1^\Delta + X_2^\Delta + ... + X_T^\Delta}$$

where X is a Lévy process with characteristic function $\psi_\Delta(u) \equiv \ln\Big(\mathbb{E}[e^{iuX_T^\Delta}]\Big)$.
Let us assume $Z_k^\Delta \equiv m\Delta + X_k^\Delta$ and its characteristic function ϕ, then

$$\sum_{k=1}^{T} S_{\Delta k} = e^{Z_1^\Delta}\Big(1 + e^{Z_1^\Delta}\Big(...\Big(1 + e^{Z_T^\Delta}\Big)\Big)\Big).$$

Staring from $L_T^\Delta \equiv e^{Z_1^\Delta}$ and recursively we can define

$$L_k^\Delta = e^{Z_k^\Delta}(1 + L_{k+1}^\Delta), k = T-1, .., 1$$

then

$$A_T^\Delta = S_0(1 + L_1^\Delta)/(T+1)$$

and the moments of the arithmetic average then can be computed as follows:

$$\mathbb{E}[(A_T^\Delta)^n] = (\frac{S_0}{T+1})^n \sum_{j=0}^{n}\binom{n}{j}\mathbb{E}[(L_T^\Delta)^j] \tag{3.2.10}$$

where $\mathbb{E}[(L_T^\Delta)^j]$ can be computed by the following recursion

$$\mathbb{E}[(L_k^\Delta)^n] = e^{nm\Delta}\sum_{q=0}^{n}\binom{n}{q}\mathbb{E}[(L_{k+1}^\Delta)^q] \tag{3.2.11}$$

with initial condition $\mathbb{E}[(L_T^\Delta)^n] = \phi_{X^\Delta}(-in)$. If we consider $B_1^\Delta = \ln(L_1^\Delta)$ and its density f_{B_1}, the call price of the option is

$$\mathbb{E}[C_f x^a(K,T)] = e^{-rT}\int_\gamma^\infty \Big(\frac{S_0}{T+1}(1+e^x) - K\Big) f_{B_1}(x)dx \tag{3.2.12}$$

where $\gamma = \ln(K(T+1)/S_0 - 1)$. Since, density of $B_k = \ln(L_k^\Delta) = Z_k^\Delta + \ln(1 + L_{k+1}^\Delta)$ is the convolution of the density $f_{Z_k^\Delta}$ and $\ln(1 + e^{B_{k+1}^\Delta})$. The function $f_{B_j}(x)$ can be computed recursively

$$f_{B_k}(x) = \int_{-\infty}^{\infty} f_Z^\Delta(x - \ln(e^y + 1)) f_{B_{k+1}}(y)dy, k = T-1, ..., 1 \tag{3.2.13}$$

with initial condition $f_{B_T} = f_Z^\Delta$. Now this integral can be approximated using an M-point quadrature formula as

$$f_{B_k}(x) = \sum_{j=1}^{M} w_j \phi(x - \ln(e^{y_j} + 1)) f_{B_{k+1}}(y_j) \tag{3.2.14}$$

3.2.4 Pricing Asian Options based on Fourier Cosine Expansions

The starting point for pricing plain vanilla European options proposed in [96] is the risk-neutral option valuation formula (the discounted expected payoff approach), i.e.,

$$v(x,t_0) = e^{-r(T-t_0)} \int_{-\infty}^{\infty} v(y,T) f(y|x) dy,$$

where $v(y,T)$ pay-off function for European options and $f(y|x)$ is the transitional density function. The density function can be approximated on a truncated domain $[a,b]$ by Fourier Cosine series enpension with N terms given a conditional characteristic function $\phi(u;x)$, is given by

$$f(y|x) \approx \frac{2}{b-a} \sum_{k=0}^{N-1} \Re\Big(\phi\Big(\frac{k\pi}{b-a};x\Big) \exp\Big(-i\frac{ak\pi}{b-a}\Big)\Big) \cos\Big(k\pi\frac{y-a}{b-a}\Big)$$

and the payoff function of an arithmetic Asian option is

$$v(S,T) \equiv g(S) = \begin{cases} \max\Big(\frac{1}{M+1}\sum_{j=0}^{M} S_j - K, 0\Big) \text{for a call} \\ \max\Big(K - \frac{1}{M+1}\sum_{j=0}^{M} S_j, 0\Big) \qquad \text{for a put} \end{cases}$$

For arithmetic Asian options, the characteristic function of the arithmetic mean will be derived recursively by Fourier cosine expansions and Clenshaw–Curtis quadrature. We first explain the recursion procedure for recovering the characteristic function of the arithmetic mean value of the underlying. We denote

$$R_j := \log\big(\frac{S_j}{S_{j-1}}\big), j = 1,.., M$$

Let us define a stochastic process, Y_j for $j = 2, ..., M$ and $Y_1 = R_M$ such that

$$Y_j := R_{M+1-j} + Z_{j-1}$$

where $Z_j = \log(1 + \exp(Y_{j-1}))$. Then Y_j can be written as

$$Y_j = \log\Big(\frac{S_{M-j+1}}{S_{M-j}} + \frac{S_{M-j+2}}{S_{M-j}} + ... + \frac{S_M}{S_{M-j}}\Big)$$

and we have

$$\frac{1}{M+1}\sum_{j=0}^{M} S_j = \frac{(1+\exp(Y_M))S_0}{M+1}$$

Now the arithmetic Asian option value can be redefined as

$$v(x_0, t_0) = e^{-r\Delta t} \int_{-\infty}^{\infty} v(y, T) f_{Y_M}(y) dy$$

where

$$v(y, T) \equiv g(S) = \begin{cases} \max\Big(\dfrac{1}{M+1}(S_0(1+\exp(y))) - K\Big)^+ \text{for a call} \\ \max\Big(K - \dfrac{1}{M+1}(S_0(1+\exp(y)))\Big)^+ \text{for a put} \end{cases}$$

and $f_{Y_M}(y)$ is the density function of Y_M. To recover the characteristic function of Y_M, i.e., $\phi_{Y_M}(u)$, we start with Y_1, for which the characteristic function known as $\phi_{Y_1}(u) = \phi_R(u)$. Then, at time steps $t_j, j = 2, ..., M, \phi_{Y_j}(u)$ can be recovered in terms of $\phi_{Y_{j-1}}(u)$. This implies that, $\forall j, R_{M+1-j}$ and Z_{j-1} are independent, we have

$$\phi_{Y_j}(u) = \phi_{R_{M+1-j}(u)} \phi_R(u) \phi_{Z_{j-1}}(u).$$

Now from the definition of characteristic function, we have

$$\phi_{Z_{j-1}}(u) = \mathbb{E}[e^{iu \log(1+\exp(Y_{j-1}))}] = \int_{-\infty}^{\infty} (e^x + 1)^{iu} f_{Y_{j-1}}(x) dx$$

We first truncate the integration range and applying Fourier cosine series expansion to approximate the characteristic function,

$$\phi_{Z_{j-1}}(u) = \int_a^b (e^x + 1)^{iu} f_{Y_{j-1}}(x) dx$$

and applying the Fourier cosine expansion to approximate $f_{Y_{j-1}}(x)$,

$$\phi_{Z_{j-1}}(u) = \frac{2}{b-a} \sum_{l=0}^{N-1} \Re\Big(\phi_{Y_{j-1}}\Big(\frac{l\pi}{b-a}\Big) \exp\Big(-ia\frac{l\pi}{b-a}\Big)\Big) \times \int_a^b (e^x + 1)^{iu} \cos\Big((x-a)\frac{l\pi}{b-a}\Big) dx$$

3.3 Proposed New Method

In this section, we present two main theorems related to arithmetic Asian options. Let $A(t) = \int_0^t S_u \, du$. Then A is an increasing continuous process and thus has no Brownian component. In general, there are four different types of arithmetic Asian options according to the payoff functions. Let T be the time of expiry of the option. For *fixed strike* (E) call and put Asian options payoffs are given by $\big(\frac{1}{T}A - E\big)^+$ and $\big(E - \frac{1}{T}A\big)^+$ respectively. For *floating strike* call and put Asian options the payoffs are given by $\big(S - \frac{1}{T}A\big)^+$ and $\big(\frac{1}{T}A - S\big)^+$ respectively. In this section, we develop a technique for pricing floating strike put Asian options. Option pricing for other Asian options can be done by very similar procedure. We first show that the price of the Asian option is given by a PIDE.

Theorem 3.3.1. *The price of floating put Asian option $P(t,S,A)$, where the stock-price dynamics is described by* (2.5.1)*, is given by*

$$\frac{\partial P}{\partial t} + rS\frac{\partial P}{\partial S} + S\frac{\partial P}{\partial A} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} - rP$$
$$+ \int_{\mathbb{R}} \left[P(t, Se^x, A) - P(t,S,A) - S(e^x - 1)\frac{\partial P}{\partial S} \right] \nu_Q(dx) = 0, \tag{3.3.1}$$

with the final condition

$$P(T,S,A) = \left(\frac{A}{T} - S\right)^+. \tag{3.3.2}$$

Proof. Let us assume there exists a smooth continuous function for put price given by $P : [0,T] \times [0,\infty] \times [0,\infty] \to \mathbb{R}$, $P \in \mathcal{C}^{1,2,2}$. Under an equivalent martingale measure $\mathbb{Q}$, the put price of floating Asian option type can be written as

$$P(t,S,A) = e^{-r(T-t)}\mathrm{E}^Q\Big[\Big(\frac{A(T)}{T} - S(T)\Big)^+ |\mathcal{F}_t\Big].$$

From the dynamics of the stock price under $\mathbf{Q}$ is given by (2.5.5). We define the continuous part and jump of $S(t)$ by

$$dS^c(t) = S(t-)rdt + \sigma S(t-)dW(t),$$

and

$$\Delta S = S(t) - S(t-),$$

respectively.
By definition $dA(t) = S(t)dt$ is a stochastic process which is strictly increasing and therefore can not have Brownian part. Also, it has no jumps. Thus $A(t)$ is a stochastic process without Brownian and Jump terms. Therefore $\Delta A = 0$ or $A(t) = A(t-)$, for $t \geq 0$.

Define

$$\tilde{P}(t,S,A) = e^{r(T-t)}P(t,S,A). \tag{3.3.3}$$

The random variables A and S are Levy processes. Since all square-integrable Levy processes are semimartingales [(2.3.14)], we can apply two-dimensional Itô formula for semimartingale on $\tilde{P}$, hence

$$\begin{aligned} d\tilde{P}(t,S,A) = e^{r(T-t)}\Big\{ &\frac{\partial P(t,S,A)}{\partial S}dS + \frac{\partial P(t,S,A)}{\partial A}dA + \frac{\partial P}{\partial t}dt - rPdt. \\ &+ \frac{1}{2}\frac{\partial^2 P(t,S,A)}{\partial S^2}d[S^c,S^c](t) + \frac{1}{2}\frac{\partial^2 P(t,S,A)}{\partial A^2}d[A^c,A^c](t) \\ &+ \frac{\partial^2 P(t,S,A)}{\partial S \partial A}d[S^c,A^c](t) + P(t,S(t),A(t)) - P(t,S(t-),A(t-)) \\ &- \frac{\partial P(t,S,A)}{\partial S}\Delta S - \frac{\partial P(t,S,A)}{\partial A}\Delta A\Big\}, \end{aligned} \tag{3.3.4}$$

where $[\cdot,\cdot]$ is the quadratic variation of two stochastic processes.

For present case, we obtain

$$d[S^c, S^c](t) = \sigma^2 S^2 d[W(t), W(t)](t) = \sigma^2 S^2 dt,$$

and since there is no Brownian part for A, we have

$$d[A^c, A^c](t) = 0 \text{ and } d[S^c, A^c](t) = 0.$$

Thus, (3.3.4) gives

$$d\tilde{P}(t, S, A) = a(t)dt + dM(t),$$

where

$$a(t) = e^{r(T-t)} \left[\frac{\partial P}{\partial t} + rS\frac{\partial P}{\partial S} + S\frac{\partial P}{\partial A} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} - rP \right.$$
$$\left. + \int_{\mathbb{R}} \left(P(t, Se^x, A) - P(t, S, A) - S(e^x - 1)\frac{\partial P}{\partial S} \right) \nu_Q(dx) \right]$$

and

$$dM(t) = e^{r(T-t)}\sigma S(t)\frac{\partial P}{\partial S}dW(t) + e^{r(T-t)} \int_{\mathbb{R}} \left(P(t, Se^x, M) - P(t, S, M)\right) \tilde{N}(dt, dx).$$

Clearly $M(t)$ is a martingale. By construction $\tilde{P}(t, S, A) = \mathrm{E}\big[H(S(t))|\mathcal{F}_t\big]$ is a martingale and hence both $M(t)$ and $\tilde{P}$ are martingales with respect to $\mathbb{Q}$. Therefore $\tilde{P}(t, S, A) - M(t)$ is also a martingale. But $\tilde{P}(t, S, A) - M(t) = \int_0^t a(s)ds$ is a continuous process with finite variation. Therefore we must have $a(t) = 0$ a. s..

Thus, we obtain (3.3.1) for $0 \le t \le T$ and $0 < S < \infty$. Note that $P(t, S, A) \to 0$ as $S \to \infty$ and satisfies the boundary conditions $P(T, S, A) = \left(\frac{A}{T} - S\right)^+$. □

The next theorem gives a computational procedure of the option price P in Theorem (3.3.1). We assume that the value of A is bounded by $A_{\max}$ for $0 \le t \le T$.

Theorem 3.3.2. *The price of Asian floating put option $P(t, S, A)$ is given by*

$$P(t, S, A) = \frac{e^{-k\log\left(S - \frac{1}{T}A + \frac{1}{T}A_{\max}\right)}}{\pi} Re\Big(\int_0^{\infty} e^{i\eta\log\left(S - \frac{1}{T}A + \frac{1}{T}A_{\max}\right)} \Big[H(\eta)e^{(T-t)\psi(\eta)}\Big] d\eta \Big), \tag{3.3.5}$$

where $k > 0$ is a constant,

$$\Psi(\eta) = -\frac{1}{2}\sigma^2\eta^2 + i\eta\left(r - \frac{1}{T} - \frac{\sigma^2}{2} - k\sigma^2\right) + \left(-k\left(r - \frac{1}{T} - \frac{\sigma^2}{2}\right) + \frac{1}{2}\sigma^2 k^2 - r\right) + I(\eta), \tag{3.3.6}$$

where

$$I(\eta) = \int_{\mathbb{R}} \left[e^{-i\eta x + kx} - 1 - (e^{-x} - 1)(-k + i\eta)\right] w_Q(-x)\, dx, \tag{3.3.7}$$

and

$$H(\eta) = \frac{\left(\frac{A_{\max}}{T}\right)^{k+1-i\eta}}{(k - i\eta)(k + 1 - i\eta)}. \tag{3.3.8}$$

In the expression of $I(\eta)$, $w_Q(x)$ is the Lévy density as defined at the beginning of this section.

Proof. We consider a special form of solution given by $P(t, S, A) = G(t, S - \frac{A}{T})$. Then $\frac{\partial P}{\partial t} = \frac{\partial G}{\partial t}, S\frac{\partial P}{\partial S} = S\frac{\partial G}{\partial S}, S\frac{\partial P}{\partial A} = -\frac{S}{T}\frac{\partial G}{\partial S}$, $S^2\frac{\partial^2 P}{\partial S^2} = S^2\frac{\partial^2 G}{\partial S^2}$. Therefore the PIDE (3.3.1) is given by

$$\begin{aligned}&\frac{\partial G}{\partial t} + rS\frac{\partial G}{\partial S} - \frac{S}{T}\frac{\partial G}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} - rG \\ &\quad + \int_{\mathbb{R}} \left[G(t, Se^x - \frac{A}{T}) - G(t, S - \frac{A}{T}) - S(e^x - 1)\frac{\partial G}{\partial S}\right] \nu_Q(dx) = 0.\end{aligned}$$

Due to the form of the solution, it is sufficient to take A to be fixed and equal to $A_{\max}$. Let the corresponding values of S be denoted as S^*. For S and A with $S^* - \frac{1}{T}A_{\max} = S - \frac{1}{T}A$, clearly $G(t, S - \frac{A}{T}) = G(t, S^* - \frac{A_{\max}}{T})$.

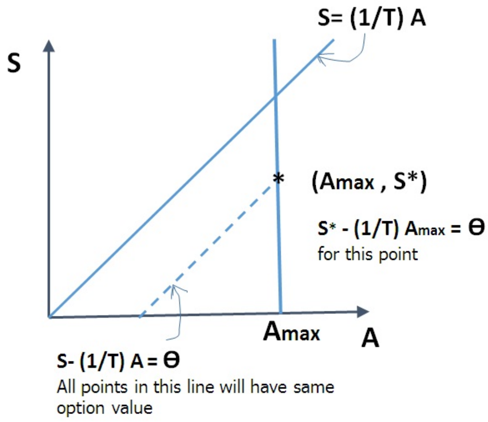

Figure 3.1: Straight line on which G has identical value for any $\theta \in \mathbb{R}$.

Let $S^* = e^z$, and therefore, $S^*\frac{\partial G}{\partial S^*} = \frac{\partial G}{\partial z}, S^{*2}\frac{\partial^2 G}{\partial S^{*2}} = \frac{\partial^2 G}{\partial z^2} - \frac{\partial G}{\partial z}$. Thus the previous expression (with $A = A_{\max}$ and x replaced by $-x$) becomes

$$\begin{aligned}&\frac{\partial G}{\partial t} + \left(r - \frac{1}{T} - \frac{1}{2}\sigma^2\right)\frac{\partial G}{\partial z} + \frac{1}{2}\sigma^2\frac{\partial^2 G}{\partial z^2} - rG \\ &\quad + \int_{\mathbb{R}} \left[G(t, e^{z-x} - \frac{A_{\max}}{T}) - G(t, e^z - \frac{A_{\max}}{T}) - (e^{-x} - 1)\frac{\partial G}{\partial z}\right] w_Q(-x)\, dx = 0.\end{aligned}$$

Let us define a new function $f(t,z) = G(t, e^z - \frac{A_{\max}}{T})$. We obtain

$$\frac{\partial f}{\partial t} + \left(r - \frac{1}{T} - \frac{1}{2}\sigma^2\right)\frac{\partial f}{\partial z} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial z^2} - rf \\ + \int_{\mathbb{R}} \left[f(t,z-x) - f(t,z) - (e^{-x}-1)\frac{\partial f}{\partial z}\right] w_Q(-x)\,dx = 0. \tag{3.3.9}$$

The function $f(t,z) \to 0$ as $z \to \infty$ but reaches a finite value for $z \to -\infty$. Hence, it is not a square-integrable function on $\mathbb{R}$. We consider the Fourier transform of the modified function

$$g(t,z) = e^{kz} f(t,z), \quad k > 0. \tag{3.3.10}$$

For a range of positive values of k, we expect $g(t,z)$ is square integrable over $\mathbb{R}$. Later in this section we find a range/choice of k for different processes.

Clearly $\frac{\partial f}{\partial t} = e^{-kz}\frac{\partial g}{\partial t}$, $\frac{\partial f}{\partial z} = e^{-kz}\left(\frac{\partial g}{\partial z} - kg\right)$ and $\frac{\partial^2 f}{\partial z^2} = e^{-kz}\left(\frac{\partial^2 f}{\partial z^2} - 2k\frac{\partial g}{\partial z} + k^2 g\right)$. Therefore (3.3.9) becomes

$$\frac{\partial g}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 g}{\partial z^2} + \left(r - \frac{1}{T} - \frac{\sigma^2}{2} - k\sigma^2\right)\frac{\partial g}{\partial z} + \left(-k\left(r - \frac{1}{T} - \frac{\sigma^2}{2}\right) + \frac{1}{2}\sigma^2 k^2 - r\right) g \\ + \int_{\mathbb{R}} \left[e^{kx} g(t,z-x) - g(t,z) - (e^{-x}-1)\left(\frac{\partial g}{\partial z} - kg\right)\right] w_Q(-x)dx = 0. \tag{3.3.11}$$

Denote the Fourier transform of $g(t,z)$ with respect to z as $\hat{g}(t,\eta)$. Applying the Fourier transform with respect to z to (3.3.11) and using convolution and other properties (described in Appendix), we obtain

$$\frac{\partial \hat{g}(t,\eta)}{\partial t} + \Psi(\eta)\hat{g}(t,\eta) = 0, \tag{3.3.12}$$

where $\Psi(\eta)$ is given by (3.3.6).

The solution of (3.3.12) is given by

$$\hat{g}(t,\eta) = H(\eta)e^{(T-t)\Psi(\eta)}, \tag{3.3.13}$$

where $H(\eta) = \hat{g}(T,\eta)$ can be computed as

$$H(\eta) = \frac{\left(\frac{A_{\max}}{T}\right)^{k+1-i\eta}}{(k-i\eta)(k+1-i\eta)}.$$

Therefore, the put option price can be written as,

$$P(t,S^*,A_{\max}) = G(t,S^* - \frac{A_{\max}}{T}) = e^{-k\log(S^*)}\mathcal{F}^{-1}\left[H(\eta)e^{(T-t)\psi(\eta)}\right]\Bigg|_{\eta\leftrightarrow\log S^*}. \tag{3.3.14}$$

Thus,

$$P(t,S^*,A_{\max}) = \frac{e^{-k\log(S^*)}}{2\pi}\Big(\int_{-\infty}^{\infty} e^{i\eta\log(S^*)}\left[H(\eta)e^{(T-t)\psi(\eta)}\right]d\eta\Big) \\ = \frac{e^{-k\log(S^*)}}{\pi} Re\Big(\int_{0}^{\infty} e^{i\eta\log(S^*)}\left[H(\eta)e^{(T-t)\psi(\eta)}\right]d\eta\Big).$$

The second equality holds because $g(t, z)$ is real, which implies for $\hat{g}(t, \eta)$ the imaginary part is odd and the even part is even. Hence, in general, for S and A satisfying $S - \frac{1}{T}A = S^* - \frac{1}{T}A_{\max}$ (see Figure 3.1), the put price $P(t, S, A)$ is given by (3.3.5). $\square$

It is clear from Theorem 3.3.5 that the fixed k plays an important role for the computation of the price of Asian options. We proceed to find appropriate ranges of this constant for different processes. It is clear from (3.3.7) that k must satisfy $\int_{\mathbb{R}} e^{kx} w_Q(-x)dx < \infty$. Using this we can find the value of k for different processes such as the normal inverse Gaussian (NIG), CGMY, Meixner processes. We take the Föllmer-Schweizer martingale measure and assume (2.5.8).

For the NIG process with $\alpha > \beta > 0$ and $\delta > 0$, the Lévy density is

$$w(x) = e^{\beta x}\frac{\delta\alpha}{\pi|x|}K_1(\alpha|x|) = \frac{\delta}{\sqrt{2\pi\alpha}}e^{\beta x-\alpha|x|}\left[|x|^{-\frac{3}{2}} + \frac{3}{8}|x|^{-\frac{5}{2}} - \frac{15}{128}|x|^{-\frac{7}{2}} + \frac{315}{3072}|x|^{-\frac{9}{2}} + \dots\right],$$

where $K_\lambda(x)$ is the modified Bessel process of the third kind with index λ. Thus

$$w(-x) = e^{-\beta x}\frac{\delta\alpha}{\pi|x|}K_1(\alpha|x|) = \frac{\delta}{\sqrt{2\pi\alpha}}e^{-\beta x-\alpha|x|}\left[|x|^{-\frac{3}{2}} + \frac{3}{8}|x|^{-\frac{5}{2}} - \frac{15}{128}|x|^{-\frac{7}{2}} + \frac{315}{3072}|x|^{-\frac{9}{2}} + \dots\right].$$

Hence, the integral $\int_{\mathbb{R}} e^{kx} w_Q(-x)dx < \infty$ for $k < (\alpha + \beta)$. Therefore for NIG the condition on k is given by

$$(\alpha + \beta) > k > 0.$$

For the CGMY process the Lévy measure is

$$w(x) = C\frac{e^{-Mx}}{x^{1+Y}}\mathbb{1}_{x>0} + C\frac{e^{Gx}}{|x|^{1+Y}}\mathbb{1}_{x<0} \text{ when } C, G, M > 0 \text{ and } Y < 1.$$

Hence, the integral $\int_{\mathbb{R}} e^{kx} w_Q(-x)dx < \infty$ when $k < M$. Therefore for CGMY the condition on k is given by $M > k > 0$.

For the Meixner process the Lévy measure is

$$\begin{aligned}
w(x) &= \delta x \exp\left(\frac{\beta}{\alpha}x\right)\left[\sinh\left(\frac{\pi x}{\alpha}\right)\right]^{-1} \quad (\text{when } -\pi < \beta < \pi,\ \alpha, \delta > 0) \\
&= \delta x e^{\beta j x} 2 e^{-\frac{\pi}{\alpha}x}\sum_{j=0}^{\infty} e^{-2j.\frac{\pi}{\alpha}x} \\
&= 2\delta x \sum_{j=0}^{\infty} e^{-\left[(2j+1)\pi-\beta\right]\frac{x}{\alpha}} \\
w(-x) &= 2\delta x \sum_{j=0}^{\infty} e^{-\left[(2j+1)+\beta\right]\frac{x}{\alpha}}.
\end{aligned}$$

Hence, the integral $\int_{\mathbb{R}} e^{kx} w_Q(-x)dx < \infty$ when $0 < k < \frac{\pi+\beta}{\alpha}$.

We conclude this section with the computation of *sensitivities* for floating type put Asian options.

Theorem 3.3.3. *The pricing Sensitivities of the Asian floating type put option are given by*

$$\begin{aligned}
\Delta(t,S,A) &= \frac{1}{\pi\left(S-\frac{1}{T}A+\frac{1}{T}A_{\max}\right)^{k+1}} \\
&\quad Re\Big(\int_0^\infty e^{i\eta\log\left(S-\frac{1}{T}A+\frac{1}{T}A_{\max}\right)}\Big[(i\eta-k)H(\eta)e^{\Psi(\eta)(T-t)}\Big]d\eta\Big), \\
\Gamma(t,S,A) &= \frac{1}{\pi\left(S-\frac{1}{T}A+\frac{1}{T}A_{\max}\right)^{k+2}}\times \\
&\quad Re\Big(\int_0^\infty e^{i\eta\log\left(S-\frac{1}{T}A+\frac{1}{T}A_{\max}\right)}\Big[(i\eta-k)(i\eta-k-1)H(\eta)e^{\Psi(\eta)(T-t)}\Big]d\eta\Big), \\
\Theta(t,S,A) &= -\frac{1}{\pi\left(S-\frac{1}{T}A+\frac{1}{T}A_{\max}\right)^{k}} \\
&\quad Re\Big(\int_0^\infty e^{i\eta\log\left(S-\frac{1}{T}A+\frac{1}{T}A_{\max}\right)}\Big[\Psi(\eta)H(\eta)e^{\Psi(\eta)(T-t)}\Big]d\eta\Big),
\end{aligned}$$

where $H(\eta)$ *and* $\Psi(\eta)$ *are given by* (5.3.3) *and* (3.3.6) *respectively.*

Proof. In our case, we have considered only the sensitivities which are given by

$$\Delta(t,S,A)=\frac{\partial P}{\partial S};\quad \Gamma(t,S,A)=\frac{\partial^2 P}{\partial S^2};\quad \Theta(t,S,A)=\frac{\partial P}{\partial t}.$$

Hence the results follow from various differentiations of (3.3.5). □

3.4 Numerical Results

As the Lévy market is incomplete, it may have more than one or mathematically infinite number of equivalent martingale measures. We describe a method to determine an unique Lévy measure ν from the market data by using *non-parametric calibration.* Given observed market prices of options, we follow the non-parametric approach for identification of the Lévy measure.

Let us consider the (observed) market prices $P^*(T_i,S_i,A_i)$, $i=1,\ldots,n$, for a set of liquid put options. The objective is to find constants ν such that

$$P^\nu(T_i,S_i,A_i)=P^*(T_i,S_i,A_i), \tag{3.4.1}$$

where P^ν is the option price computed for parameters ν.

The popular approach to non-linear least squares is

$$(\nu^*)=\arg\inf_\nu \sum_{i=1}^{N}\{P^\nu(T_i,S_i,A_i)-P^*(T_i,S_i,A_i)\}^2.$$

The usual formulations of the inverse problems via nonlinear least squares are ill-posed and in [34] a regularization method is proposed on relative entropy. In [34] the calibration problem is reformulated into problem of finding a risk-neutral jump-diffusion model that reproduces the observed option prices and has the smallest possible relative entropy with respect to a chosen prior model. In the calibration for the present chapter we use this technique.

Algorithm 1 describes the procedure for computing the put price of the Asian options. We test our model against the Microsoft US Equity Asian put option prices between 02/11/2014 to 05/27/2014 available from Bloomberg. The dataset consists of 66 mid prices of a Asian options on Microsoft at the close of the market during the above time period. The stock prices are graphed in the Figure 3.2. On the start date the mid-stock price is \$37.545, average is \$37.3783, and effective strike is \$37.5835. Interest rate on the start date is $r = 0.02336$, and option price is \$0.77. On the start date Delta (%) is -42.4, Gamma (%) is 5.6252, Vega is 0.04, Theta is -0.01, Rho is 0 and Gearing is 48.78.

In Table 3.1 we provide the calibration results for the given data set with three different processes (as Lévy density)- NIG, CGMY and Meixner. Figure 3.3 shows the model put prices for these three processes. Resulting calibration fitting with market prices are provided in in Figure 3.4. The parameters estimated from the calibration procedure resemble the current market view on the asset. In Table 3.2 we provide the APE, AAE, ARPE and RMSE for the calibration using NIG, CGMY and Meixner processes.

Algorithm 1 Algorithms for computing the Asian put option

Input: The put option of floating type available from market with stock price between any start time t ending maturity time T.

Output: Put price at any future time after t, $P(t, S, A)$.

1: {**Step 1**}

Estimate the Lévy measure ν with the help of Least Square Estimation described above.

2: {**Step 2**}

Define the function $\hat{P}(t, \eta, A_{\max}) = H(\eta)e^{(T-t)\psi(\eta)}$ where

$\Psi(\eta) = -\frac{1}{2}\sigma^2\eta^2 + i\eta\left(r - \frac{1}{T} - \frac{\sigma^2}{2} - k\sigma^2\right) + \left(-k\left(r - \frac{1}{T} - \frac{\sigma^2}{2}\right) + \frac{1}{2}\sigma^2 k^2 - r\right) + I(\eta)$ with

$H(\eta) = \frac{\left(\frac{A_{\max}}{T}\right)^{k+1-i\eta}}{(k-i\eta)(k+1-i\eta)}$ and $I(\eta) = \int_{\mathbb{R}} \left[e^{-i\eta x + kx} - 1 - (e^{-x} - 1)(-k + i\eta)\right] w_Q(-x)\, dx.$

The integral $I(\eta)$ can be calculated using Clenshaw-Curtis rule described in the Appendix.

3: {**Step 3**}

Calculate $\hat{P}(t, \eta_j, A_{\max})$ for all $\eta_j = (j-1)\triangle\eta, j = 1, \ldots, L.$

4: {**Step 4**}

Calculate the put price using Fourier inverse transform of the put price described in the Appendix. $P(t, S^*, A_{\max}) \approx \frac{e^{-k\log(S^*)}}{\pi}\sum_{j=1}^{L} e^{i\eta_j \log(S^*)}\hat{P}(t, \eta_j, A_{\max})\triangle\eta.$

5: {**Step 5**}

Extend the solution to $P(t, S, A) = P(t, S^*, A_{\max})$, for $S - \frac{1}{T}A = S^* - \frac{1}{T}A_{\max}$.

For *goodness of fit* of the calibration, we use the absolute percentage error (APE), the average absolute error (AAE), the average relative percentage error (ARPE) and the root-mean-square error (RMSE) given by the following formulas.

$$APE = \frac{1}{\text{mean option price}} \sum_{options} \frac{|\text{market price} - \text{model price}|}{\text{number of options}},$$

$$AAE = \sum_{options} \frac{|\text{market price} - \text{model price}|}{\text{number of options}},$$

$$ARPE = \frac{1}{\text{number option price}} \sum_{options} \frac{|\text{market price} - \text{model price}|}{\text{number of options}},$$

$$RMSE = \sqrt{\sum_{options} \frac{(\text{market price} - \text{model price})^2}{\text{number of options}}}.$$

Table 3.1: Estimated Parameters for Lévy processes

Model	Parameters			
NIG	$\alpha = 2.5589$	$\beta = -0.2367$	$\delta = 6.3449$	
CGMY	$C = 8.3234$	$G = 1.4281$	$M = 3.0832$	$Y = 0.0100$
Meixner	$\alpha = 2.2806$	$\beta = 2.1259$	$\delta = 0.7163$	

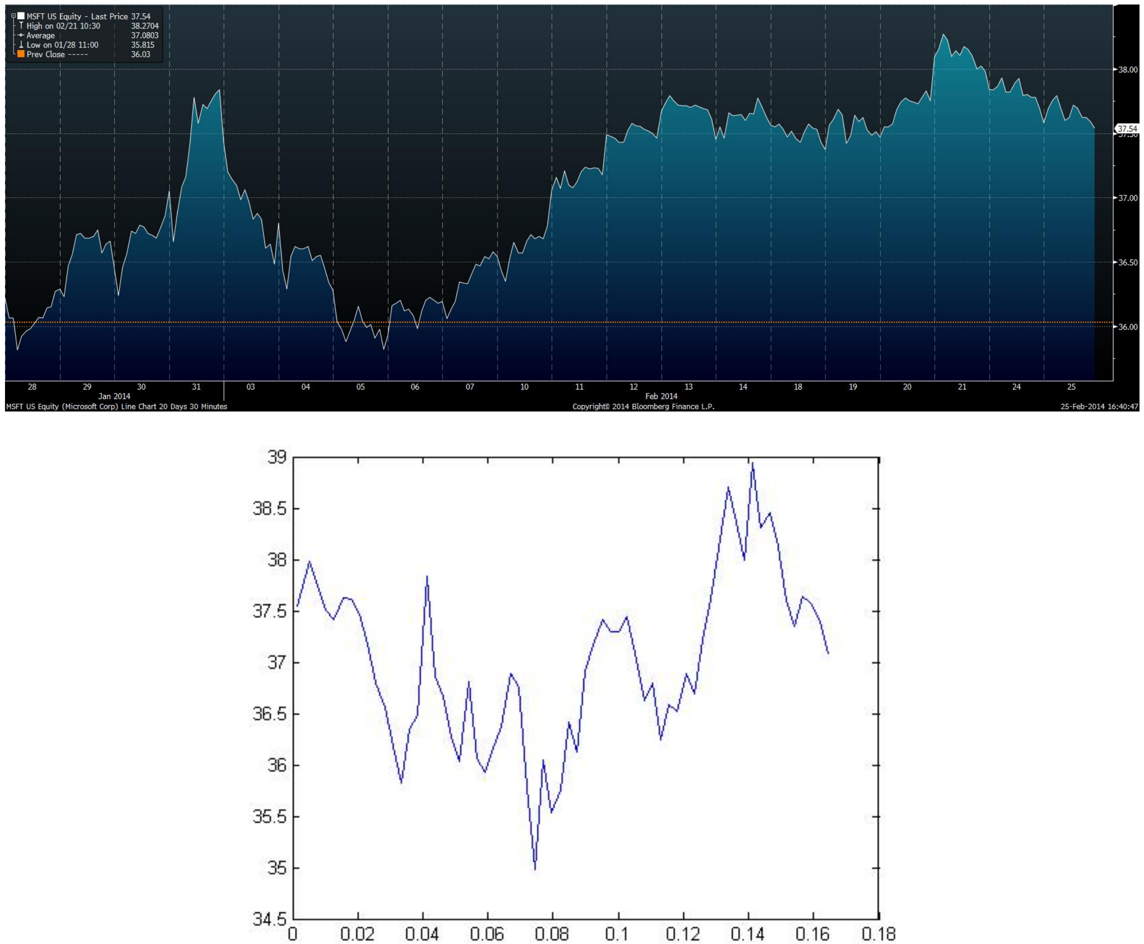

Figure 3.2: Top: Microdoft US Equity Price from Bloomberg, Bottom: Microdoft US Equity Price between 2/11/2014 to 05/27/2014 (obtained from the previous graph).

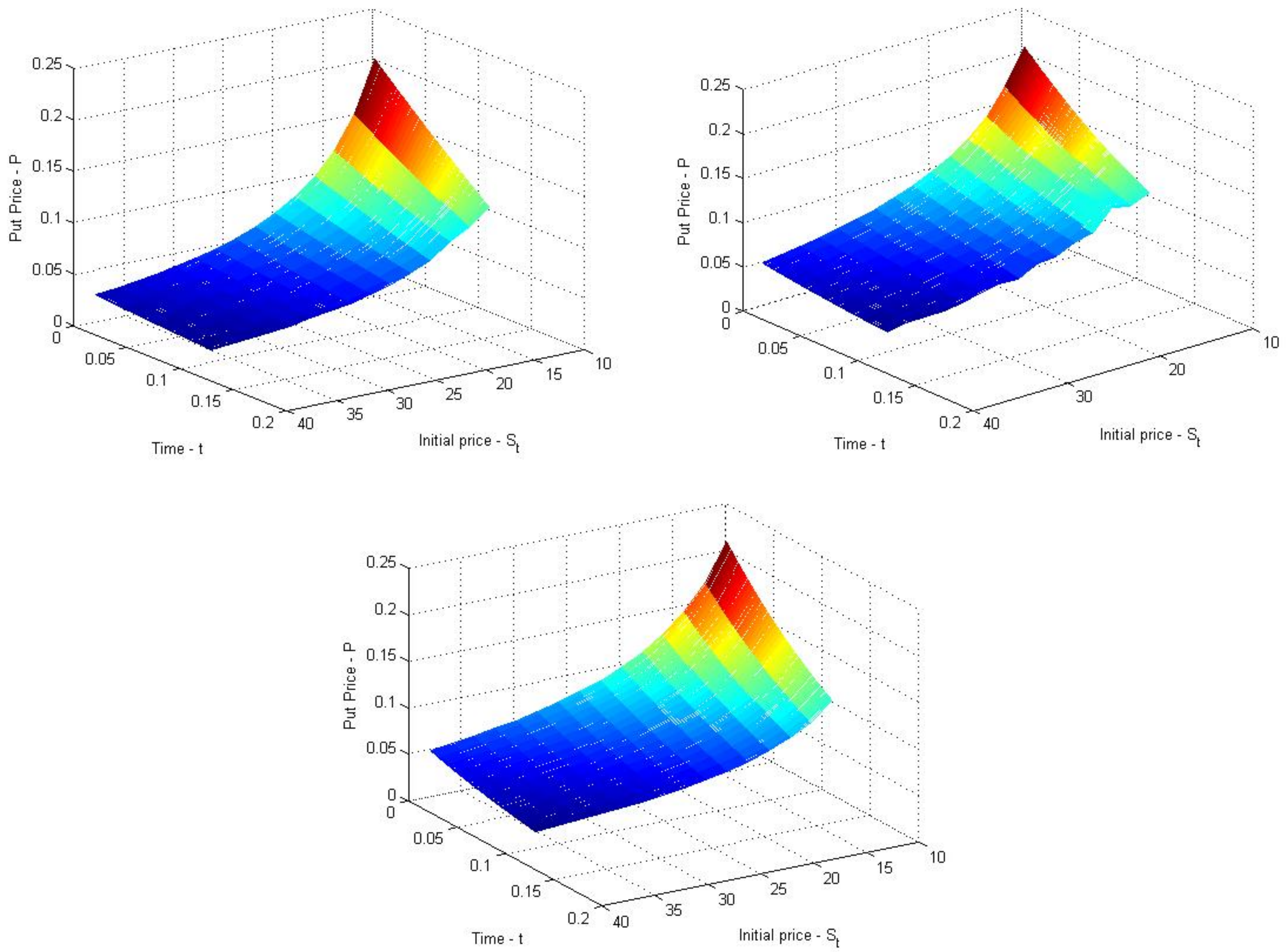

Figure 3.3: Asian Put Option with CGMY (top-left), NIG (top-right) and Meixner (bottom) processes.

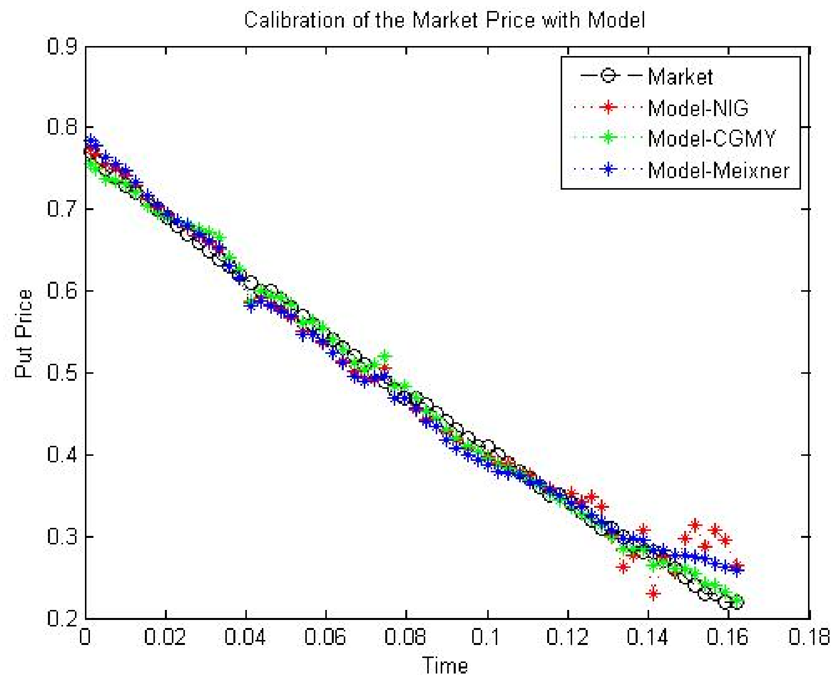

Figure 3.4: Calibration of proposed model with NIG, CGMY and Meixner process.

Table 3.2: Performance of above Lévy Models

Model	APE	AAE	RMSE	ARPE
NIG	3.4351	1.6419	0.2052	0.0257
CGMY	1.6279	0.7781	0.0973	0.0122
Meixner	2.9650	1.4172	0.1771	0.0221

It is clear from Table 3.2 that the model and calibration performance with respect to the market data is very high, as all the goodness of fit measures, especially RMSE and ARPE, are reasonably small. Comparing with [82] (Chapter 6 and Chapter 7), it is clear that such goodness of fit measures are significantly better than that of calibration of pricing vanilla options using Black-Scholes and different Lévy model techniques.

Using the above estimated parameters, in Table 3.3, Table 3.4 and Table 3.5, we calculate various changes of the put price of Microsoft Asian option of arithmetic type and various changes of the *greeks*. For the following tables the Asian Put Option has initial stock value S and maturity $T = 60$ days.

Table 3.3: Put price change with time to maturity, stock price and its average

t	S	Avg	NIG(α, β, δ) Put	CGMY(C, G, M, Y) Put	Meixner(α, β, δ) Put
10.31	36.56	36.7963	0.6472	0.6491	0.6465
20.63	36.0550	36.2956	0.5307	0.5418	0.5266
30.00	35.7600	35.7165	0.4387	0.4454	0.4318
40.31	36.800	36.8555	0.3753	0.3697	0.3688

Table 3.4: Change of Delta & Gamma over Stock price change

S	NIG (α, β, δ)		CGMY (C, G, M, Y)		Meixner (α, β, δ)	
	Delta	Gamma	Delta	Gamma	Delta	Gamma
35.53	-0.0022	-0.0153	0.0128	-0.0054	0.1016	-0.6155
36.41	-0.0128	-0.0293	0.0120	-0.0144	-0.2658	-0.8611
37.29	0.0023	-0.0347	0.0109	-0.0070	0.6201	-2.7739
38.705	0.0167	-0.1438	0.0081	-0.0226	4.2739	-34.2603

Table 3.5: Change of Theta over Time to expire

t	NIG (α, β, γ) Theta	CGMY (C, G, M, Y) Theta	Meixner (α, β, δ) Theta
10.310	-227.7086	-17.6052	-85.4738
20.6300	-30.1431	-6.3451	-35.7301
30.000	-42.0272	-3.4475	-17.8429
40.310	24.7487	-0.7626	773.8078
50.6300	57.1492	-1.8635	6141

3.5 Conclusion

The characterization of Asian option prices in terms of solutions of PIDE allows to use efficient numerical methods for pricing floating put Asian options in the presence of jumps. We analytically develop an expression which solves that option pricing PIDE. In this procedure the distribution of the arithmetic average of stock prices is not required for calculation. This is an interesting improvement over the existing methods in literature. We develop an efficient technique to compute prices of the option and greeks. We determine the modified Lévy process under suitable equivalent martingale measure and subsequently use the Fourier inversion technique to numerically calculate the put prices and greeks. We have shown that it is possible to compute the Asian option of arithmetic type with the Lévy process having infinite number of negative and positive jumps without finding the distribution of arithmetic average of stock price. Numerical result shows that a very good market calibration can be obtained with this proposed method.

Chapter 4

Barrier Options

4.1 Introduction

Barrier options are derivatives with a pay-off that depends on whether a reference entity has crossed a certain boundary. Common examples are the knock-in and knock-out call and put options that are activated or deactivated when the underlying crosses a specified Barrier-level. Barrier and Barrier-type options belong to the most widely traded exotic options in the financial markets.

A class of models that has been shown to be capable of generating a good fit of observed call and put option price data is formed by the infinite activity Lévy models, such as normal inverse Gaussian, CGMY and Meixner. This class of models has been extensively studied and we refer for background and further references to the book by [82]. In this chapter, we consider Barrier options driven by Lévy processes with infinite activity. This class contains many of the Lévy models used in financial modelling as the fore-mentioned ones.

Several approaches have been proposed during the last few years. The calculation of first-passage distributions and Barrier option prices in (specific) Lévy models has been investigated in a number of papers. In [56], the authors proposed a Laplace transformed based approach to compute the prices and greeks of Barrier options for a class of Lévy process with Wiener-Hopf factorisation. The authors of [51] calculated prices and deltas of double Barrier options under the Black-Scholes model. For spectrally one-sided Lévy processes with a Gaussian component [80] derived a method to evaluate first-passage distributions. The authors of [61],[68],[85] followed a transform approach to obtain Barrier prices for a jump-diffusion with exponential jumps. In the setting of infinite activity Lévy processes with jumps in two directions Cont and [24] investigated discretisation of the associated integro-differential equations. In [14], the author employed Fourier methods to investigate Barrier option prices for Lévy processes of regular exponential type. These approaches are based on exponential Lévy process with a risk neutral measure considering a complete market, involving extremely complex techniques and applicable for a specific class of Lévy process.

Summarizing all the issues in the previous work, we find a few challenges in pricing the Barrier option under Lévy processes. First of all, the Lévy market is incomplete and more than one measure exists leading to multiple prices for a single contract and hedging is not possible. Therefore, the pricing model requires the selection of the correct measure from the market and finding market price of risk with the help of market price available by calibration method with better goodness of fit. Secondly, as the distribution of the underlying stock prices is unknown, in general no explicit analytical expression is available. Finally, it is also difficult to derive a closed form expression of the contract. Our model is proposed to take care of all the challenges. The approach first developed a PIDE for pricing and solved it using Mellin transform and its inverse. The advantage of our model is that it has a closed form expression of the Mellin transform applicable for any class of Lévy processes and the standard inverse Mellin transform can be applied to construct prices. In [28], the author proposed a similar method for Asian options of arithmetic type but used Fourier transform instead of Mellin transform.

The organization of different sections in this chapter is as follows. Section 2 recalls some basic facts about exponential Lévy processes and provides a model used in this chapter. Section 4.3 derives the partial integro-differential equation (PIDE) for the option pricing of Barrier options. It also provides a pricing formula in terms of the inverse Mellin transform and numerical results are provided in Section 4.4.

4.2 Benchmark Models

4.2.1 Monte Carlo Simulation

The method described in[82] simulates m paths of stock prices process and calculate for each path the value of the payoff function $V_i, i = 1, ..., m$. Then the Monte-Carlo estimate of the expected value of the payoff is

$$\tilde{V} = \frac{1}{m}\sum_{n=1}^{m} V_i \tag{4.2.1}$$

The final option price is then obtained by discounting this estimate $e^{-rT}V$ and standard error (SE) of the estimate is given by

$$\tilde{V} = \sqrt{\frac{1}{(m-1)^2}\sum_{n=1}^{m}(\hat{V} - V_i)^2} \tag{4.2.2}$$

The Lévy process simulated by two methods namely 1) Compound-Poisson approximation and 2) Independent Poisson process. In Compound-Poisson approximation method, the Lévy measure $\nu(dx)$ is discretized by partitioning $\mathbb{R}\backslash[-\varepsilon\ \varepsilon]$ where $0 < \varepsilon < 1$, such that $a_0 < a_1, .. < a_k = -\varepsilon, \varepsilon = a_{k+1}, a_{k+2}, .., < a_{d+1}$. The independent Poisson process $N_t^{(i)}$ for the interval ,$[a_{i-1}, a_i), 1 \leq i \leq k$ and $[a_i, a_{i+1}), k+1 \leq i \leq d$, with intensity λ_i. The simulation method done by replacing small jumps with their expected values and the process X^d consists of Brownian motion $W = W_t, t \geq 0$ and d independent Poission process $N^{(i)} = N_t^{(i)}, t \geq 0, i = 1, .., d$ with parameter λ_i. Then

$$
\begin{aligned}
X_t^{(d)} =& \gamma t + \tilde{\sigma} W_t + \sum_{i=1}^{d} c_i (N_t^{(i)} - \lambda_i t 1_{|c_i|<1}) \\
\lambda_i =& \nu([a_{i-1}, a_i)) \text{ for } i \leq k \\
=& \nu([a_i, a_{i+1})) \text{ for } k+1 \leq i \leq d \\
c_i^2 \lambda_i =& \int_{a_{i-1}}^{a_i^-} x^2 \nu(dx) \text{ for } i \leq k \\
=& \int_{a_i}^{a_{i+1}^-} x^2 \nu(dx) \text{ for } k+1 \leq i \leq d
\end{aligned}
$$

and the variation of Brownian process is approximated by small jumps and can be written as

$$
\tilde{\sigma}^2 = \sigma^2 + \sigma^2(\varepsilon) \text{ where } \sigma^2(\varepsilon) = \int_{|x|<\varepsilon} x^{\nu}(dx)
$$

The intervals $[a_{i-1}, a_i), 1 \leq i \leq k$ and $[a_i, a_{i+1}), k+1 \leq i \leq d$ considered different types like Equally Spaced, Equally Weighted and Inverse Linear Boundaries. For Equally Spaced interval $|a_{i-1} - a_i|$ is kept fixed, whereas for Equally Weighted interval the Lévy measure for positive $\nu([a_i - a_{i+1}))$ for $1 \leq i \leq k$ and negative jumps $\nu([a_{i-1} - a_i))$ for $k+1 \leq i \leq d$ are kept fixed. The Inverse Linear Boundary case intervals defined as $a_{i-1} = -\alpha i^{-1}$ and $a_{2k+2-i} = \alpha i^{-1}$. The price of the Barrier option is calculated by Variance Reduction of Control Variates as follows

1. Consider G and $H$are the payoff function of the exotic option and vanila options which are positively correlated.We want to calculate the expected value of G given a control variate H.

2. Define a new payoff function for some real number

$$
\hat{G}(S_t, 0 \leq t \leq T) = G(S_t, 0 \leq t \leq T) + b(H(S_t, 0 \leq t \leq T) - E[H])
$$

.

3. Sample n number of paths of the stock price $S = S_t, 0 \leq t \leq T$ and calculate payoffs $g_i = G(S_t, 0 \leq t \leq T)$ and $h_i = HS_t, 0 \leq t \leq T$ for each path i.

4. Estimate of b is $\hat{b} = \frac{1}{n}\Big(\sum_{i=1}^{n} g_i h_i - E[H] \sum_{i=1}^{n} g_i \Big)$

5. Expected payoff of the exotic option is $\hat{g} = \frac{1}{n}\Big(\sum_{i=1}^{n} g_i - \hat{b}(h_i - E[H]) \Big)$

4.2.2 The Wiener–Hopf factorization and Pricing Expression

Let X be a Lévy process with exponent $\psi(\xi)$ where it admits analytic continuation from $\mathbb{R}$ into strip $\xi \in (\lambda_-, \lambda_+)$. Fix λ, and set $q = i\lambda + r$. If $\Re q > 0$, then the factorization of $a(\lambda, \xi) = q + \psi(\xi)$ can be done for any Lévy process [see, e.g., Theorem 45.2 in [88]] though without the explicit formulas for the factors $a_-(\lambda, \xi)$ and $a_+(\lambda, \xi)$. The explicit formulas are derived for any RLPE in [14],[15].

We assume that the riskless rate $r > 0$ is constant and, under a risk-neutral measure $\mathbb{Q}$, X is an RLPE of exponential type $[\lambda_-, \lambda_+]$, where $\lambda_- < -1 < 0 < \lambda_+$.

We consider "out" options with barrier H, without a rebate; the terminal payoffs are of the form

$$G(S_T) = (S_T^\beta - K)^+$$

, where $0 \leq \beta < -\lambda_-$, which includes payoffs for calls and "power calls" [we denote the price of such a contract by $W^\beta_{;call}(K, H, T; S, T)$], or of the form

$$G(S_T) = (K - S_T^\beta)+$$

,where $-\lambda_- < \beta$, which includes payoffs for puts and "power puts" [the notation used is $W^\beta_{;put}(K, H, T; S, T)$]. We also consider barrier contracts with payoffs $G(S_T) = S_T^\beta$; in the down-and-out case, the price is denoted by $W^\beta_{do}(H, T; S, t)$,and in the up-and-out case, by $W^\beta_{uo}(H, T; S, t)$. (More general payoffs can also be considered.)In addition to W^β_{do} and W^β_{uo}, we have to consider separately six cases of standard (power) barrier options:
(i) down-and-out call option $W^\beta_{do}; call(K, H, T; S, T)$, in the case $K \leq H^\beta$;
(ii) up-and-out put option $W^\beta_{uo}; put(K, H, T; S, T)$, in the case $K \leq H^\beta$;
(iii) down-and-out put option $W^\beta_{do}; put(K, H, T; S, T)$, in the case $K > H^\beta$;
(iv) up-and-out call option $W^\beta_{uo}; call(K, H, T; S, T)$, in the case $K < H^\beta$;
(v) down-and-out call option $W^\beta_{do}; call(K, H, T; S, T)$, in the case $K > H^\beta$;
(vi) down-and-out put option $W^\beta_{do}; put(K, H, T; S, T)$, in the case $K < H^\beta$.

The following theorem for pricing expressions have been developed and discussed in [14].

Theorem 4.2.1. *(a) Let $\beta < -\lambda_-$. Then, for $t < T$ and $S > H$,*

$$W^\beta_{do}(H, T; S, t) = \frac{H^\beta}{(2\pi)^2 a_+(\lambda, -i\beta)} \int_{-\infty+i\sigma}^{\infty+i\sigma} \int_{-\infty+i\omega_-}^{\infty+i\omega_-} \times \frac{\exp(i[\lambda(T-\tau) + \xi \ln(S/H)]) d\xi d\lambda}{a_-(\lambda, \xi)(i\xi - \beta)}$$

where $\omega_- \in (\lambda_-, -\beta)$ is arbitrary.

(b) Let $\beta < -\lambda_+$. Then, for $t < T$ and $S < H$,

$$W^{\beta}_{uo}(H,T;S,t) = \frac{H^{\beta}}{(2\pi)^2 a_+(\lambda,-i\beta)} \int_{-\infty+i\sigma}^{\infty+i\sigma} \int_{-\infty+i\omega_-}^{\infty+i\omega_-}$$
$$\times \frac{\exp(i[\lambda(T-\tau)+\xi\ln(S/H)])d\xi d\lambda}{a_+(\lambda,\xi)(-i\xi+\beta)}$$

where $\omega_+ \in (-\beta,\lambda_+)$ is arbitrary.

Theorem 4.2.2. *(a) Let $K > H^{\beta}$, $\beta \geq 0$ and $k_+ > 0$. Then, for $t < T$ and $S > H$,*

$$W^{\beta}_{do;put}(H,T;S,t) = \frac{1}{(2\pi)^3 i} \int_{-\infty+i\sigma}^{\infty+i\sigma} \int_{-\infty+i\gamma_1}^{\infty+i\gamma_1} \int_{-\infty+i\gamma}^{\infty+i\gamma} e^{i[\lambda(T-\tau)+\xi\ln(S/H)]}$$
$$\times \frac{\hat{g}_+(K,H,\beta;\zeta)d\zeta\ d\xi d\lambda}{a_-(\lambda,\xi)(\xi-\zeta)a_+(\lambda,\xi)}$$

where a negative $\sigma \leq \sigma_0$ and $\lambda_- < \gamma_1 < \gamma < \lambda_+$ are arbitrary.

(b) Let $K < H^{\beta}$, $0 < \beta \leq -\lambda+$ and $k_- > 0$. Then, for $t < T$ and $S < H$,

$$W^{\beta}_{uo;put}(H,T;S,t) = \frac{1}{(2\pi)^3 i} \int_{-\infty+i\sigma}^{\infty+i\sigma} \int_{-\infty+i\gamma_1}^{\infty+i\gamma_1} \int_{-\infty+i\gamma}^{\infty+i\gamma} e^{i[\lambda(T-\tau)+\xi\ln(S/H)]}$$
$$\times \frac{\hat{g}_-(K,H,\beta;\zeta)d\zeta\ d\xi d\lambda}{a_+(\lambda,\xi)(\xi-\zeta)a_-(\lambda,\xi)}$$

where a negative $\sigma \leq \sigma_0$ and $\lambda_- < \gamma_1 < \gamma < \lambda_+$ are arbitrary.

4.2.3 The Wiener–Hopf factorization, First Passage and Laplace transform

Wiener-Hopf factorization formula used in probability

$$E[e^{i\xi X_T}] = E[e^{i\xi \overline{X}_T}]E[e^{i\xi \underline{X}_T}]$$

, $\forall \xi \in \mathbb{R}$ where $T \sim Exp\ q$, and $\overline{X}_T = \sup_{0\leq s\leq t} X_s$ and $\underline{X}_T = \inf_{0\leq s\leq t} X_s$ are the supremum and infimum of processes. Then

$$\phi_+(\xi) = qE\Big[\int_0^{\infty} e^{-qt}e^{i\xi\overline{X}_t}\Big] = E[e^{i\xi\overline{X}_T}]$$

$$\phi_-(\xi) = qE\Big[\int_0^{\infty} e^{-qt}e^{i\xi\underline{X}_t}\Big] = E[e^{i\xi\underline{X}_T}]$$

and

$$\frac{q}{q+\psi(\xi)} = \phi_+(\xi)\phi_-(\xi)$$

.

Remark 4.2.3. *Let $q = i\lambda + r$ and*

$$\frac{q}{q+\psi(\xi)} = \phi_+(\xi)\phi_-(\xi)$$

be Wiener-Hopf factorization formula used in probability. Then for regular Lévy process $\phi_+(\xi) = \phi_+(\lambda, \xi), \phi_-(\xi) = \phi_-(\lambda, \xi)$ and their inverses are polynomially bounded w.r.t. ξ in the corresponding half-planes [this is proved in Boyarchenko and Levendorskiĭ (2002)];

An analytical expression of Laplace Transform for Pricing Barrier option under a class of Lévy process with hyper-exponential jumps derived in [56]. The method follows Wiener-Hopf factorisation and First Passage time for finding the distribution of Maxima and Minima of the underlying Lévy process. The analytical expression for Knock-out and knock-in put options are derived thereafter.

The Lévy density of hyper-exponential jump-diffusion (HEJD) given by

$$k(x) = \lambda^+ \sum_{i=1}^{n^+} p_i^+ \alpha_i^+ e^{-\alpha_i^+ x} \mathbb{1}_{x>0} + \sum_{j=1}^{n^-} p_j^- \alpha_j^- e^{-\alpha_j^- x} \mathbb{1}_{x<0} \tag{4.2.3}$$

where $\alpha_i^\pm, \lambda^\pm, p_i^\pm > 0$ with $\sum_{i=1}^{n^\pm} p_i^\pm = 1$. The corresponding Lévy exponent is

$$\psi(u) = \mu u i - \frac{\sigma^2}{2}u^2 + \lambda^+ \sum_{i=1}^{n^+} p_i^+ \alpha_i^+ \Big(\frac{\alpha_i^+}{\alpha_i^+ - ui} - 1\Big) + \lambda^- \sum_{j=1}^{n^-} p_j^- \alpha_j^- \Big(\frac{\alpha_j^-}{\alpha_j^- + ui} - 1\Big) \tag{4.2.4}$$

Let $\rho_i^+ = \rho_i^+(q), i = 1, ..., m^+$ and $\rho_j^- = \rho_j^-(q), j = 1, ..., m^-$ are the roots of $\psi(-is) = q$ with positive and negative real parts respectively. Then, Wiener–Hopf factors can be written as

$$\phi_q^+(u) = \frac{\prod_{i=1}^{n^+}\Big(1 - \frac{ui}{\alpha_i^+}\Big)}{\prod_{i=1}^{m^+}\Big(1 - \frac{ui}{\rho_i^+(q)}\Big)} \qquad \phi_q^-(u) = \frac{\prod_{j=1}^{n^-}\Big(1 - \frac{ui}{\alpha_j^-}\Big)}{\prod_{j=1}^{m^-}\Big(1 - \frac{ui}{\rho_j^-(q)}\Big)} \tag{4.2.5}$$

Then Laplace Transform of the distribution of $\overline{X}_t = \sup_{0\le s\le t} X_s$ and $\underline{X}_t = \inf_{0\le s\le t} X_s$ are given by

$$\int_0^\infty e^{-qt} P(\overline{X}_t \le z) = \frac{1}{q}\Big(1 - \sum_{i=1}^{m^+} A_i^+ e^{-\rho_i^+(q)z}\Big) \tag{4.2.6}$$

$$\int_0^\infty e^{-qt} P(-\underline{X}_t \le z) = \frac{1}{q}\Big(1 - \sum_{j=1}^{m^-} A_j^- e^{-\rho_j^-(q)z}\Big) \tag{4.2.7}$$

where $z \ge 0$ and the coefficients A_i^+ and A_j^- are given by

$$A^+(u) = \frac{\prod_{\nu=1}^{n^+}\Big(1 - \frac{\rho_i^+(q)}{\alpha_\nu^+}\Big)}{\prod_{\nu=1,\nu\ne i}^{m^+}\Big(1 - \frac{\rho_i^+(q)}{\rho_\nu^+(q)}\Big)} \qquad A^-(u) = \frac{\prod_{\nu=1}^{n^-}\Big(1 - \frac{\rho_j^-(q)}{\alpha_\nu^-}\Big)}{\prod_{\nu=1,\nu\ne j}^{m^-}\Big(1 - \frac{\rho_j^-(q)}{\rho_\nu^-(q)}\Big)} \tag{4.2.8}$$

The Laplace transform of down-and-out put $DOP(T) = DOP(T; S0, K, H)$ with level H, maturity T and strike K is given by the following proposition

Proposition 4.2.4. *Let $q > 0$ and $h = \log(H/S_0)$, $l = \log(K/S_0)$, then Laplace transform of down-and-out put $DOP(T)$ is*

$$D\hat{OP}(p) = \frac{1}{q+r} KC^{(0)}(l,h) - \frac{S_0}{q+r} C^{(1)}(l,h)$$

, where

$$C^{(b)}(l,h) = \sum_{j=1}^{k^-} \sum_{i=1}^{k^+} \frac{\rho_i^+(-\rho_j^-)}{(\rho_j^- - \rho_i^+)(b - \rho_i^+)} A_i^+ A_j^- (e^{(b-\rho_i^+)l + (\rho_i^+ - \rho_j^-)h} - e^{(b-\rho_j^-)})$$

$$+ \Big(1 - \sum_{i=1}^{k^+} \frac{b}{b - \rho_i^+} A_i^+\Big) \sum_{j=1}^{k^-} A_j^- \frac{-\rho_j^-}{\rho_j^- - b} (e^{(b-\rho_i^+)l} - e^{(b-\rho_j^-)})$$

when $h < l < 0$,

$$C^{(b)}(l,h) = \sum_{j=1}^{k^-} \sum_{i=1}^{k^+} \frac{\rho_i^+(-\rho_j^-)}{(\rho_j^- - \rho_i^+)(b - \rho_i^+)} A_i^+ A_j^- e^{(b-\rho_j^-)} (e^{(\rho_i^+ - \rho_j^-)h} - 1)$$

$$+ \Big(1 - \sum_{i=1}^{k^+} \frac{b}{b - \rho_i^+} A_i^+\Big) \Big(1 + \sum_{j=1}^{k^-} A_j^- \Big(\frac{-\rho_j^-}{\rho_j^- - b} e^{(b-\rho_j^-)h} - \frac{b}{\rho_j^- - b}\Big)\Big)$$

$$+ \Big(1 - \sum_{j=1}^{k^-} A_j^-\Big) \sum_{i=1}^{k^+} \frac{\rho_i^+}{b - \rho_i^+} A_i^+ e^{(b-\rho_i^+)l}$$

with $\rho_i^+ = \rho_i^+(q+r)$ and $\rho_j^- = \rho_j^-(q+r)$ when $h < 0 < l$,

4.3 Proposed New Method

In this section, we present two main theorems related to single Barrier options. Let S be the stock price and B is a fixed single Barrier. In general, there are four different types of Barrier options according to the payoff functions. Let T be the time of expiry of the option. For *fixed strike* (K) call and put Up-And-Out Barrier options payoffs are given by $(S-K)^+, 0 \le S \le B$ and $(K-S)^+, 0 \le S \le B$ respectively. For *fixed strike* call and put Down-And-Out Barrier options the payoffs are given by $(S-K)^+, B \le S$ and $(K-S)^+, B \le S$ respectively. In this section, we develop a technique for pricing fixed strike call for both Up-And-Out and Down-And-Out options. Option pricing for other type options can be done by a very similar procedure. We first show that the price of the both Up-And-Out and Down-And-Out Barrier option is given by a PIDE.

Theorem 4.3.1. *The price of Up-And-Out and Down-And-Out Barrier call option $C(t, S(t))$, where the stock-price dynamics is described by Equation* (2.5.1)*, is given by*

$$\frac{\partial C(t,S)}{\partial t} + rS\frac{\partial C}{\partial S}(t,S) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}(t,S) - rC(t,S)$$
$$+ \int_{\mathbb{R}} \nu_Q(dx)\left[C(t,Se^x) - C(t,S) - S(e^x-1)\frac{\partial C}{\partial S}(t,S)\right] = 0 \tag{4.3.1}$$

with final condition

$$C(T,S) = (S-K)^+, 0 \le S \le B \text{ for Up-And-Out option} \tag{4.3.2}$$
$$= (S-K)^+, B \le S < \infty \text{ for Down-And-Out option} \tag{4.3.3}$$

Proof. Let us assume there exists a smooth continuous function for call price given by $C : [0,T] \times [0,\infty] \to \mathbb{R}$, $C \in \mathcal{C}^{1,2}$. Under an equivalent martingale measure $\mathbb{Q}$. Under an equivalent martingale measure $\mathbb{Q}$, the Up-And-Out and Down-And-Out Barrier call option can be written as

$$C(t,S(t)) = e^{-r(T-t)}\mathrm{E}_{\mathcal{Q}}\Big[H(S_T)|\mathcal{F}_t\Big]$$

where

$$\begin{aligned} H(S_T) &= \Big(S(T)-K\Big)^+ \mathbb{1}_{S(T)\le B} \text{ for Up-And-Out option} \\ &= \Big(S(T)-K\Big)^+ \mathbb{1}_{S(T)\ge B} \text{ for Down-And-Out option} \end{aligned}$$

From the dynamics of the stock price under **Q** is given by Equation (2.5.5). We define the continuous part and jump of $S(t)$ by

$$dS^c(t) = S(t-)rdt + \sigma S(t-)dW(t)$$

and

$$\Delta S = S(t) - S(t-)$$

respectively.
The continuous part of S(t) is defined to be

$$dS^c(t) = rS(t)dt + \sigma S(t)dW(t)$$

Since all square-integrable Levy processes are semimartingales[(2.3.14)], we can apply Itô formula for semimartingale on $\tilde{C}$. Let us consider $S(t) = S$ and $\widetilde{C}(t,S(t)) = e^{r(T-t)}C(t,S(t))$ and if we can apply It$\hat{o}$'s formula to this function,

$$\begin{aligned} d\widetilde{C}(t,S(t)) &= e^{r(T-t)}\Bigg[\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} - rC \\ &\quad + \int_{\mathbb{R}}\left(C(t,Se^x) - C(t,S) - (e^x-1)S\frac{\partial C}{\partial S}\right)\nu_Q(dx)\Bigg]dt \\ &\quad + e^{r(T-t)}\frac{\partial C}{\partial S}\sigma S dW(t) \\ &\quad + e^{r(T-t)}\int_{\mathbb{R}}\Big\{C(t,Se^x) - C(t,S)\Big\}\widetilde{N}(dt,dx) \\ &= a(t)dt + dM(t) \end{aligned}$$

where

$$
\begin{aligned}
a(t) &= e^{r(T-t)}\left[\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 C}{\partial S^2} - rC\right.\\
&+ \left.\int_{\mathbb{R}}\left(C(t,Se^x) - C(t,S) - S(e^x-1)\frac{\partial C}{\partial S}\right)\nu_Q(dx)\right]
\end{aligned}
$$

and

$$
dM(t) = e^{r(T-t)}\frac{\partial C}{\partial S}\sigma S dW(t) + e^{r(T-t)}\int_{\mathbb{R}}\Big\{C(t,Se^x) - C(t,S)\Big\}\widetilde{N}(dt,dx)
$$

Clearly, $M(t)$ is a Martingale. By construction $\widetilde{C}(t,S(t)) = \mathrm{E}[H(S(t))|\mathcal{F}_t]$ and $M(t)$ both are martingales, then $\widetilde{C}(t,S(t)) - M(t)$ is also a martingale. But $\widetilde{C}(t,S(t)) - M(t) = \int_0^t a(s)ds$ is a continuous process with finite variation. Thgerefore, we must have $a(t) = 0$ almost surely. Thus, we obtain the partial integro-differential equation (PIDE),

$$
\begin{aligned}
\frac{\partial C(t,S)}{\partial t} &+ rS\frac{\partial C}{\partial S}(t,S) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}(t,S) - rC(t,S)\\
&+ \int_{\mathbb{R}}\nu_Q(dx)\left[C(t,Se^x) - C(t,S) - S(e^x-1)\frac{\partial C}{\partial S}(t,S)\right] = 0
\end{aligned} \tag{4.3.4}
$$

for $0 \leq t \leq T$ and $0 < S < \infty$ and $C(t,S) \to \infty$ as $S \to \infty$ with the boundary conditions are

Up and Out Barrier Option	Down and Out Barrier Option
$C(t,0) = 0, 0 \leq t \leq T,$	$C(t,0) = 0, 0 \leq t \leq T$
$C(t,B) = 0, 0 \leq t < T$	$C(t,B) = 0, 0 \leq t < T$
$C(T,S) = (S-K)^+, 0 \leq S \leq B$	$C(T,S) = (S-K)^+, B \leq S < \infty$

□

Theorem 4.3.2. *The Mellin transform of the price of Barrier option $C(t,S(t))$ is given by*

$$
C(t,S(t)) = S\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{K}{S}\right)^{-\eta}\left[H(\eta)e^{\psi(\eta)(T-t)}\right]d\eta \tag{4.3.5}
$$

where

$$
H(\eta) = \tfrac{1}{\eta(\eta+1)} - \left[\tfrac{(K/B)^\eta}{\eta} - \tfrac{(K/B)^{\eta+1}}{\eta+1}\right]\Big\}\ \textit{for Up-And-Out option}
$$

Or

$$
H(\eta) = \left.\begin{matrix} \frac{\left(\frac{K}{B}\right)^\eta}{\eta} - \frac{\left(\frac{K}{B}\right)^{\eta+1}}{\eta+1}, & \textit{if } \frac{K}{B} \leq 1 \\ \frac{1}{\eta(\eta+1)} & \textit{if } \frac{K}{B} \geq 1 \end{matrix}\right\}\ \textit{for Down-And-Out option}
$$

with

$$\psi(\eta) = -\frac{1}{2}\sigma^2\eta(\eta+1) + r\eta + I(\eta) \tag{4.3.6}$$

and

$$I(\eta) = \int_{\mathbb{R}} \nu_Q(dx)\left[e^{(\eta+1)x} - (1+\eta)e^x + \eta\right] \tag{4.3.7}$$

Proof. Let us assume that $y = \frac{K}{S(t)}$, then

$$\begin{aligned} C(t,S) &= e^{-r(T-t)}\mathrm{E}_{\mathcal{Q}}\Big[H(S_T)|\mathcal{F}_t\Big] \\ &= S(t)f(t,y) \end{aligned}$$

where

$$\begin{aligned} f(t,y) &= \mathrm{E}_{\mathcal{Q}}\Big[(e^{Z(T)} - y)^+|\mathcal{F}_t\Big]\mathbb{1}_{y \geq \frac{K}{B}}, \text{ for Up-And-Out} \\ &= \mathrm{E}_{\mathcal{Q}}\Big[(e^{Z(T)} - y)^+|\mathcal{F}_t\Big]\mathbb{1}_{y \leq \frac{K}{B}}, \text{ for Down-And-Out} \end{aligned}$$

Using above we have as follows,

$$\begin{aligned} \frac{\partial f}{\partial t} &- ryf_y - \frac{1}{2}\sigma^2 y^2 f_{yy} \\ &+ \int_{\mathbb{R}} \nu_Q(dx)\left[e^x\left\{f\big(t, ye^{-x}\big) - f(t,y)\right\} + (e^x - 1)yf_y\right] = 0 \end{aligned} \tag{4.3.8}$$

with the following boundary conditions

(1) Up and Out Barrier Option

$$\begin{aligned} f(T,y) &= (1-y)^+, \text{ when } \infty > y \geq \frac{K}{B} \\ &= 0 \text{ else} \end{aligned}$$

(2) Down and Out Barrier Option

$$\begin{aligned} f(T,y) &= (1-y)^+, \text{ when } 0 \leq y \leq \frac{K}{B} \leq 1 \\ &= 0 \text{ else} \end{aligned}$$

Now, the Mellin transform of the PIDE, gives us,

$$\begin{aligned} \frac{d\hat{f}(t,\eta)}{dt} &+ r\eta\hat{f}(t,\eta) - \frac{1}{2}\sigma^2\eta(\eta+1)\hat{f}(t,\eta) \\ &+ \int_{\mathbb{R}} \nu_Q(dx)\left[e^{(\eta+1)x} - (\eta+1)e^x + \eta\right]\hat{f}(t,\eta) = 0 \end{aligned}$$

At boundary condition $t = T$, $\hat{f}(T,\eta) = \hat{H}(\eta)$, and we can write

$$\hat{f}(t,\eta) = \hat{H}(\eta)e^{\psi(\eta)(T-t)} \tag{4.3.9}$$

where

$$\psi(\eta) = -\frac{1}{2}\sigma^2\eta(\eta+1) + r\eta + I(\eta)$$

and

$$I(\eta) = \int_{\mathbb{R}} \nu_Q(dx)\left[e^{(\eta+1)x} - (1+\eta)e^x + \eta\right]$$

Mellin Tranform of the boundary condition $\hat{H}(\eta)$ Up-and-Out Barrier option

$$\begin{aligned}\hat{H}(\eta) = \hat{f}(T,\eta) &= \int_{K/B}^{1} (1-y)y^{\eta-1}dy \\ &= \frac{1}{\eta(\eta+1)} - \left[\frac{(K/B)^{\eta}}{\eta} - \frac{(K/B)^{\eta+1}}{\eta+1}\right]\end{aligned} \tag{4.3.10}$$

and for Down-and-Out Barrier option is

$$\begin{aligned}\hat{H}(\eta) = \hat{f}(T,\eta) &= \int_{0}^{\frac{K}{B}} (1-y)y^{\eta-1}dy \\ &= \frac{\left(\frac{K}{B}\right)^{\eta}}{\eta} - \frac{\left(\frac{K}{B}\right)^{\eta+1}}{\eta+1}, \text{ if } \frac{K}{B} \leq 1 \\ &= \int_{0}^{1} (1-y)y^{\eta-1}dy = \frac{1}{\eta(\eta+1)} \text{ if } \frac{K}{B} \geq 1\end{aligned} \tag{4.3.11}$$

Hence, we can derive the expression for Call price for the both type of options described in (4.3.5) .

□

Theorem 4.3.3. *The Mellin transform of the pricing sensitivities of Barrier option is given by*

$$\Delta(t,S(t)) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (\eta+1)\Big(\frac{K}{S}\Big)^{-\eta}\left[H(\eta)e^{\psi(\eta)(T-t)}\right]d\eta \tag{4.3.12}$$

$$\Gamma(t,S(t)) = \frac{1}{S}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \eta(\eta+1)\Big(\frac{K}{S}\Big)^{-\eta}\left[H(\eta)e^{\psi(\eta)(T-t)}\right]d\eta \tag{4.3.13}$$

$$\Theta(t,S(t)) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \psi(\eta)\Big(\frac{K}{S}\Big)^{-\eta}\left[H(\eta)e^{\psi(\eta)(T-t)}\right]d\eta \tag{4.3.14}$$

with

$$H(\eta) = \begin{cases} \frac{1}{\eta(\eta+1)} - \left[\frac{(K/B)^{\eta}}{\eta} - \frac{(K/B)^{\eta+1}}{\eta+1}\right] & \textit{for Up-And-Out option} \\ \frac{\left(\frac{K}{B}\right)^{\eta}}{\eta} - \frac{\left(\frac{K}{B}\right)^{\eta+1}}{\eta+1}, \text{ if } \frac{K}{B} \leq 1 & \textit{for Down-And-Out option} \\ \frac{1}{\eta(\eta+1)} \text{ if } \frac{K}{B} \geq 1 & \textit{for Down-And-Out option} \end{cases}$$

and

$$\psi(\eta) = -\frac{1}{2}\sigma^2\eta(\eta+1) + r\eta + I(\eta) \tag{4.3.15}$$

with

$$I(\eta) = \int_{\mathbb{R}} \nu_Q(dx)\left[e^{(\eta+1)x} - (1+\eta)e^x + \eta\right] \tag{4.3.16}$$

Proof. Since

$$C(t, S(t)) = S\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{K}{S}\right)^{-\eta}\left[H(\eta)e^{\psi(\eta)(T-t)}\right]d\eta$$

and

$$\Delta(t, S(t)) = \frac{\partial C}{\partial S}; \Gamma(t, S(t)) = \frac{\partial^2 C}{\partial S^2}; \Theta(t, S(t)) = \frac{\partial C}{\partial t}$$

By differentiating, we will have the desired result. □

4.4 Numerical Results

As the Lévy market is incomplete, there exists more than one or mathematically infinite number of equivalent martingale measures. We describe a method to determine an unique Lévy measure ν from the market data by using *non-parametric calibration*. Given observed market prices of options, we follow the non-parametric approach for identification of the Lévy measure.

Let us consider the (observed) market prices $C^*(T_i, S_i, B)$, $i = 1, \ldots, n$, for a set of liquid put options. The objective is to find constants ν such that

$$C^\nu(T_i, S_i, B) = C^*(T_i, S_i, B), \tag{4.4.1}$$

where C^ν is the option price computed for parameters ν. The popular approach to non-linear least squares is

$$(\nu^*) = \arg\inf_\nu \sum_{i=1}^{N}\{C^\nu(T_i, S_i, B) - C^*(T_i, S_i, B)\}^2$$

The usual formulations of the inverse problems via nonlinear least squares are ill-posed and in [34] a regularization method is proposed on relative entropy. In [34] the calibration problem is reformulated into problem of finding a risk-neutral jump-diffusion model that reproduces the observed option prices and has the smallest possible relative entropy with respect to a chosen prior model. In the calibration for the present chapter we use this technique. The following parameters estimated by calibration of S&P 500 options (1970 to 2001) in [39], has been considered for computing the prices

Algorithm 2 Algorithms for computing the Barrier Call option

Require: Initial time t and stock price $S(t)$, Maturity time T, Stock growth r and Volatility σ , Lévy triplet (m, k, ν) and put price available from Market.

Ensure: $C(t, S(t))$

{**Step 1**}

Estimate the Lévy triplet (m, k, ν)

$$H(\eta) = \begin{cases} \frac{1}{\eta(\eta+1)} - \left[\frac{(K/B)^{\eta}}{\eta} - \frac{(K/B)^{\eta+1}}{\eta+1}\right] & \text{for Up-And-Out option} \\ \frac{\left(\frac{K}{B}\right)^{\eta}}{\eta} - \frac{\left(\frac{K}{B}\right)^{\eta+1}}{\eta+1}. \text{ if } \frac{K}{B} \leq 1 & \text{for Down-And-Out option} \\ \frac{1}{\eta(\eta+1)} \text{ if } \frac{K}{B} \geq 1 & \text{for Down-And-Out option} \end{cases}$$

{**Step 2**}

for $n \leftarrow 1, L$ **do**

Evaluate $I(n) = \int_{\mathbb{R}} \nu_Q(dx) \left[e^{(n+1)x} - (1+n)e^x + n\right]$ using Clenshaw Curtis quadrature rule in the Appendix C taking examples of Lévy Process from Appendix B

$\psi(n) = -\frac{1}{2}\sigma^2 n(n+1) + r\eta + I(n)$

$\widetilde{C}(t, n) = H(n)e^{\psi(n)(T-t)}$

$fVal(n) = \frac{\widetilde{P}(t,n)}{2^n\Gamma(n)}$

end for

for $k \leftarrow 1, L$ **do**

temp=0

for $n \leftarrow 1, k$ **do**

$temp = temp + (-1)^{n-1}\binom{k-1}{n-1} fVal(n)$

end for$C(k) = temp$;

end for

for $k \leftarrow 1, L$ **do**

$C(t, S(t)) = C(t, S(t)) + C(k) * e^{-\frac{S}{2}} L_{k-1}\left(\frac{S}{2}\right)$;

end for

$C(t, S(t))$

Algorithm 3 describes the procedure for computing the call price of the both Down-And-Out and Up-And-Out Barrier options. We have used above calibrated parameters to plot the call price plot against the Time-to-Maturity and Initial stock price for NIG, CGMY and Meixner processes in Figures 4.1–4.3. This help us to understand how the call price changes with the change in stock price and maturity. The change of call price and sensitivities are also computed with the change of parameters such as volatility σ, Interest rate r, initial stock price S_0 and Barrier B.

In Table 4.1 we provide the calibration results for the given data set with three different processes (as Lévy density)- NIG, CGMY and Meixner. The Algorithm 3 used to compute the call price and sensitivities and result listed in Tables 5.2–4.5. This result is also generated with the change of time-to-maturity, growth and volatility of the stock for different types of Lévy process.

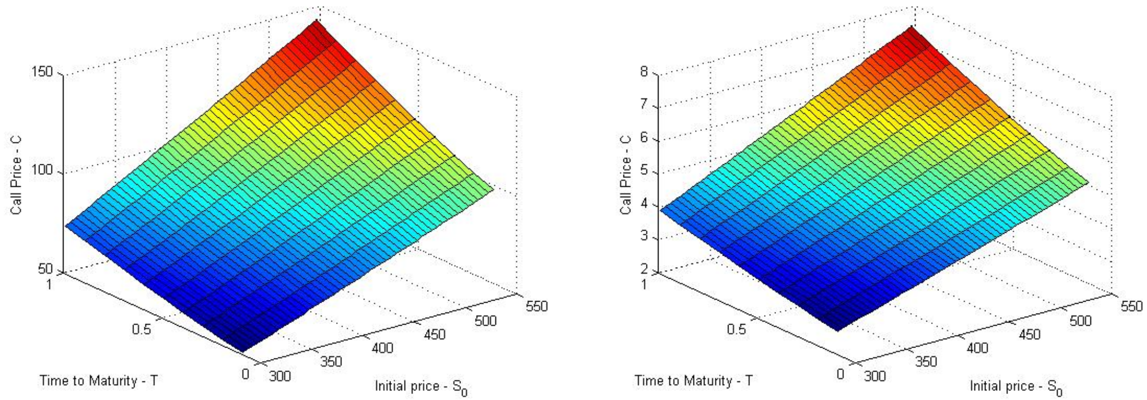

Figure 4.1: Down-And-Out and Up-And-Out call with NIG process with Stock Price $S_0 = 450$, Strike price K = 150, Barrier B = 350, $\sigma = 0.1812$, $r = 0.167$ and Time to maturity T = 1.1.

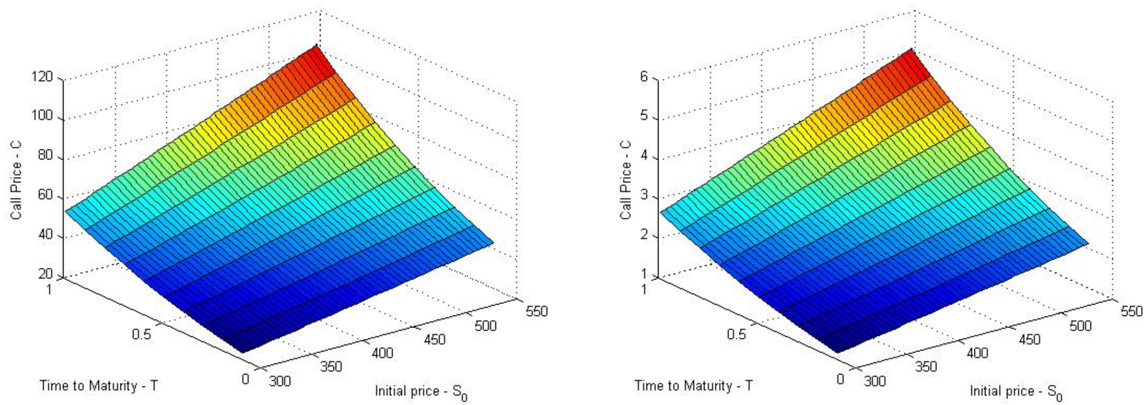

Figure 4.2: Down-And-Out and Up-And-Out call with CGMY($C = 0.0244$, $G = 0.0765$, $M = 7.5515$, $Y = 1.2945$) with Stock Price $S_0 = 450$, Strike price K = 150, Barrier B = 350, $\sigma = 0.1812$, $r = 0.167$ and Time to maturity T = 1.1.

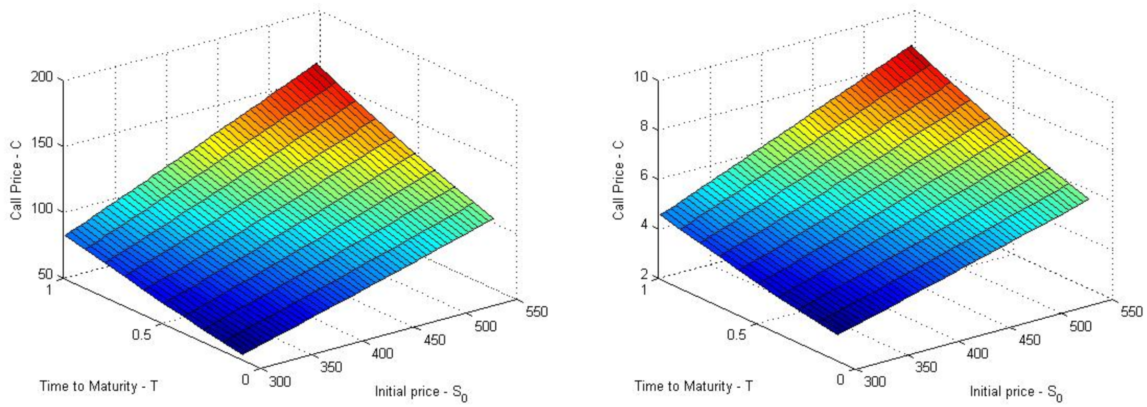

Figure 4.3: Down-And-Out and Up-And-Out call with Meixner($\alpha = 0.3977$, $\beta = -1.494$, $\delta = 0.3462$) with Stock Price $S_0 = 450$, Strike price K = 150, Barrier B = 350, $\sigma = 0.1812$, $r = 0.167$ and Time to maturity T = 1.1.

Table 4.1: Estimated parameters for Lévy processes.

Model	**Parameters**			
NIG	$\alpha = 6.1882$	$\beta = -3.8941$	$\delta = 0.1622$	
CGMY	$C = 0.0244$	$G = 0.0765$	$M = 7.5515$	$Y = 1.2945$
Meixner	$\alpha = 0.3977$	$\beta = -1.494$	$\delta = 0.3462$	

Table 4.2: Change in Call Price with different types of Lévy Process.

t	r	σ	NIG (α, β, δ) Call		CGMY (C, G, M, Y) Call		Meixner (α, β, δ) Call	
			Down-Out	**Up-Out**	**Down-Out**	**Up-Out**	**Down-Out**	**Up-Out**
1	0.167	0.5	8.8249	9.5132	8.4641	9.1207	8.8331	9.5222
	0.167	0.2	8.9626	9.6631	8.8112	9.4983	8.9689	9.6699
0.8	0.167	0.5	8.6620	9.3362	7.6423	8.2272	8.6863	9.3626
	0.167	0.2	9.0740	9.7843	8.6218	9.2924	9.0932	9.8052
0.5	0.167	0.5	8.4232	9.0767	6.5559	7.0477	8.4706	9.1282
	0.167	0.2	9.2436	9.9691	8.3454	8.9918	9.2827	10.0117

Call option with stock Initial value $S = 300$, Strike price K = 150, Barrier B = 450 and Time to maturity T = 1.1.

Table 4.3: Call Price & Sensitivities change with Barrier.

Barrier (B)	Call		Delta		Gamma		Theta	
	Down-Out	**Up-Out**	**Down-Out**	**Up-Out**	**Down-Out**	**Up-Out**	**Down-Out**	**Up-Out**
250	15.8960	3.4417	0.0410	0.0058	8.7909	1.9033	-0.6604	-0.1417
300	14.0080	5.3296	0.0374	0.0094	7.7468	2.9474	-0.5825	-0.2196
350	12.4266	6.9111	0.0342	0.0126	3.8220	6.8722	-0.5172	-0.2850

Option with Stock price $S = 350$, K = 150, $\sigma = 0.1812$, $r = 0.167$, Time to maturity T = 1.1 and NIG $(\alpha = 6.1882, \beta = -3.8941, \delta = 0.1622)$ as Lévy Process.

Table 4.4: Change of Delta and Gamma over Stock price change.

S_0	**NIG (α, β, δ)**				**CGMY (C, G, M, Y)**				**Meixner (α, β, δ)**			
	Delta		**Gamma**		**Delta**		**Gamma**		**Delta**		**Gamma**	
	Down-Out	**Up-Out**	**Down-Out**	**Up-Out**	**Down-Out**	**Up-Out**	**Down-Out**	**Up-Out**	**Down-Out**	**Up-Out**	**Down-Out**	**Up-Out**
350	0.03	0.01	6.87	3.82	0.02	0.01	4.97	4.61	0.02	0.01	5.53	5.13
400	0.03	0.01	9.09	5.05	0.02	0.01	6.58	6.09	0.03	0.01	7.32	6.78
450	0.04	0.01	11.64	6.45	0.03	0.01	8.42	7.80	0.03	0.02	9.38	8.67

Barrier Call option with Strike price K = 150, Barrier B = 350, $\sigma = 0.1812$, $r = 0.167$ and Time to maturity T = 1.1.

Table 4.5: Change of Theta over Time to expire.

t	**NIG (α, β, γ)**		**CGMY (C, G, M, Y)**		**Meixner (α, β, δ)**	
	Theta		**Theta**		**Theta**	
	Down-Out	**Up-Out**	**Down-Out**	**Up-Out**	**Down-Out**	**Up-Out**
0.4	−0.6851	−0.3763	0.7374	0.6763	−0.5246	−0.4801
0.6	−0.6753	−0.3710	0.7536	0.6910	−0.5175	−0.4736
0.8	−0.6655	−0.3657	0.7702	0.7061	−0.5104	−0.4672
1.0	−0.6560	−0.3605	0.7871	0.7215	−0.5034	−0.4608

Barrier Call option with Stock Price $S_0 = 450$, Strike price K = 150, Barrier B = 350, $\sigma = 0.1812$, $r = 0.167$ and Time to maturity T = 1.1.

The Sensitivities like Delta, Gamma and Theta of the option with respect to initial stock price S_t and t will be denoted by

$$\Delta = \frac{\partial}{\partial S}C(S, B, t); \Gamma = \frac{\partial^2}{\partial S^2}C(S, B, t); \Theta = \frac{\partial}{\partial t}C(S, B, t)$$

Using the above equations for sensitivities, we will check how the Call, Delta, Gamma & Theta changes with the change of Barrier for a specific type of Lévy process (in this case NIG) in the Table 5.3.

The Call Price and Sensitivities (Delta, Gamma and Theta) computed (Tables 4.4 and 4.5) for different types of Lévy process with its parameters.

4.5 Conclusion

In this paper, we have focused only on three types of Lévy process with infinite activity but finite moments to option pricing and compared the results. We developed an alternative techniques to compute prices and sensitives of the Barrier options. Here, we first determined the modified Lévy process under measure for incomplete market followed by development of a Partially Integro-Differential Equation and subsequently used the Mellin transform technique to get an expression for options. The expression was computed numerically with a class of Lévy process with infinite activity where distribution of the process is unknown.

Chapter 5

Look-back Options

5.1 Introduction

Look-back options are among the most popular path-dependent derivatives traded in exchanges worldwide. The payoffs of these options depend on the minimum (or maximum) asset price at the time of expiration. A standard European Look-back call (put) gives the option holder the right to buy (sell) an asset at its lowest (highest) price during the life of the option. The pricing formula for Standard (also called floating strike) Look-backs were first given by [52] in the Black-Scholes framework. Later, the fixed strike Look-backs was introduced in [36]. The discrete time approach of finding the maximum of the asset price in [6] and the continuity corrections in the Black-Scholes setting are proposed in [18]. All of these approaches proposed based on the Black-Scholes setting and risk neutral measures considering complete market.

Path dependent option such as Look-back valuation under Lévy processes has been approached by many researchers but remains a mathematical and computational challenge (See, e.g., [[84], [63], [73]]) first proposed a Laplace transform based approach on pricing path dependent option like Barrier and Look-back for Brownian model and jump-diffusion models. The Wiener-Hopf factorization method is a standard tool for pricing path-dependent options. [73] obtained formulas in terms of the Wiener-Hopf factors for the Laplace transform of continuously monitored barrier and Look-back options in general Lévy models. But approach require very complex set of computations. The probabilistic approach also taken for barrier options derived in [15] using the analytical form of the Wiener-Hopf factorization method. [62] suggested a completely new method for simulating the joint law of the maximum using Wiener–Hopf Monte Carlo technique to calculate the Look-back options. Later, [49] also presented a method for calculating Look-back options for discretely monitored price using Wiener–Hopf Technique.

The jump diffusions methods such as exponentially distributed Poisson jumps (a double-exponential jump diffusion process (DEJD) and its generalization a hyper-exponential jump-diffusion model (HEJD)) used for pricing with the help of Laplace transform method. The pricing expressions for DEJD model were given by [69] and

[63], and for double-barrier options by [86] for HEJD, see [[25], [26], [56]].

The Calculation of the Laplace transform of the price under a general Lévy process is a non-trivial problem. [16] develop a method based on Spitzer's identity to price discrete Look-back options in a general Lévy model. The methods of these papers are computationally expensive when the monitoring is frequent. In [[44], [45]], a Hilbert transform based a new computationally efficient method proposed for pricing discrete Barrier and Look-back options and presented a method of calculation of exponential moments of the discrete maximum of a Lévy process.

[22] used the fast Gaussian transform method to price discrete barrier and Look-back options in the Black-Scholes model and the Merton jump-diffusion model. As the number of monitoring times goes to infinity, discrete (barrier) Look-back options converge to continuous (barrier) Look-backs but slowly. Then, [64] developed a fast and accurate numerical method labelled Fast Wiener-Hopf factorization method (FWHF-method) for pricing continuously monitored barrier options under Lévy processes of a wide class.

An efficient approximation of the Wiener-Hopf factors based on FWHF-method is proposed for exact formula and the Fast Fourier Transform (FFT) used for computation. The method is does not require a detailed analysis of the underlying Lévy model unlike finite difference. [11] suggested an enhanced numerical realization of the FFT, which improves the convergence of the FWHF-method. [12] and later [13] developed fast and accurate techniques for calculating prices of finite lived double barrier options under regime-switching HEJD and general Lévy process. These approaches are based on exponential Lévy process involve extremely complex techniques and some are applicable for specific class of Lévy process only with high computational time.

5.2 Benchmark Models

5.2.1 Efficient pricing options with Look-back features under Lévy processes

There are several forms of the Wiener-Hopf factorization. The Wiener-Hopf factorization formula used in probability reads:

$$\mathbb{E}[e^{i\xi X_T}] = \mathbb{E}[e^{i\xi \overline{X_T}}]\,\mathbb{E}[e^{i\xi \underline{X_T}}], \forall \xi \in \mathbb{R}$$

where $T \sim Exp\, q$ is exponentially distributed random variable independent of X, and $\overline{X_t} = \sup_{0\le s\le t} X_s$ and $\underline{X_t} = \inf_{0\le s\le t} X_s$ are the supremum and infimum processes. The Wiener-Hopf factors $\phi_q^+(\xi)$ and $\phi_q^-(\xi)$ are defined as

$$\phi_q^+(\xi) = qE\Big[\int_0^\infty e^{-qt} e^{i\xi\overline{X_t}}\Big] = E\Big[e^{i\xi\overline{X_{T_q}}}\Big]$$
$$\phi_q^-(\xi) = qE\Big[\int_0^\infty e^{-qt} e^{i\xi\underline{X_t}}\Big] = E\Big[e^{i\xi\underline{X_{T_q}}}\Big]$$

If $q > 0$ is sufficiently large and $\Re(q+\psi(\xi)) > 0$ for all ξ in the strip $\Im\xi \in (\lambda_-, \lambda_+)$. Take a negative $\omega_- > \lambda_-$. Boyarchenko and Levendorskii (see eq. (3.58) and (3.60) in [15] proved that for ξ in the upper half-plane $\Im\xi > 0$,

$$\phi_q^+(\xi) = \exp\Big[\frac{1}{2\pi i}\int_{\Im\eta=\omega_-}\frac{\xi\ln(q+\psi(\eta))}{\eta(\xi-\eta)}d\eta\Big]$$

for any $\omega_- \in (\lambda_-, 0)$, and for any ξ in the lower half-plane $\Im\xi < 0$,

$$\phi_q^-(\xi) = \exp\Big[-\frac{1}{2\pi i}\int_{\Im\eta=\omega_+}\frac{\xi\ln(q+\psi(\eta))}{\eta(\xi-\eta)}d\eta\Big]$$

where $\omega_+ \in (0, \lambda_+)$.

The integrals under the exponential sign needs to be calculated at the points of the chosen grid on the line $\Im\xi = \omega$. This can be done using FFT and iFFT. For instance, for $\Im\xi = \omega > 0$ and $\omega_- \in (\lambda_-, 0)$,

$$\frac{1}{(2\pi i)}\int_{\Im\eta=\omega_-}\frac{\xi\ln(q+\psi(\eta))}{\eta(\xi-\eta)}d\eta = \xi F_{x\to\xi}\mathbb{1}_{(-\infty,0]}\frac{\ln(q+\psi(\eta))}{i\eta}$$

and for $\Im\xi = \omega < 0$ and $\omega_+ \in (0, \lambda_-)$,

$$\frac{1}{(2\pi i)}\int_{\Im\eta=\omega_+}\frac{\xi\ln(q+\psi(\eta))}{\eta(\xi-\eta)}d\eta = \xi F_{x\to\xi}\mathbb{1}_{[0,\infty)}\frac{\ln(q+\psi(\eta))}{i\eta}$$

The pricing of European floating strike Look-back options described in [65] assuming maturity T, log of stock price $X_t = \log S_t$ and its infimum $\underline{X_t}$ starts at $x = X_0$, is given by

$$V(T,x) = \mathbb{E}^x\Big[e^{-rT}(e^{X_T} - e^{\underline{X_T}})\Big]$$

The pricing approach proposed a methods based on Laplace Transform with respect to maturity T and Wiener-Hopf factorization of the option $V(T,x)$ is given by

$$\hat{V}(q,x) = \Big[\frac{1}{q+r} - \frac{\phi_{q+r}^-(q+r)}{q+r}\Big]e^x$$

The prices $V(T,x)$ can be determined by Laplace inverse transformed by the Gaver-Stehfest algorithm of prices $\hat{V}(q,x)$ where $\phi_{q+r}^-(q+r)$ can be calculated numerically by Fast Wiener-Hopf factorization method(FWHF) in [65].

5.2.2 Lookback Option Pricing Using the Fourier Transform B-spline Method

The proposed method [in [54]] is an efficient formula approximating the price of discrete look-back options based on inverting the Fourier transform using B-spline approximation and Spitzer formula for the characteristic function of the maximum. The pricing formula is given

Proposition 5.2.1. *If $M_{X_m}(v) = E_Q(e^{vX_m})$ exists for all $v \in (a, b)$, with $a < 1/2$ and $b > 1$ then, using the Fourier transform, $C(T, R)$ is given by*

$$C(T,R) = \phi_{X_m}(-i) - \frac{\sqrt{R}}{2\pi}\int_{-\infty}^{\infty} \Re\Big[\phi_{X_m}(-u - i/2)R^{iu}\Big]\frac{du}{u^2 + 1/4}$$

where $R > 0$, and ϕ_{X_m} is the characteristic function of the maximum log-return, $X_m = \max_{j=1,\ldots,m} L_{j^*+j}$, over all monitoring points from current time $t(T_0 \leq t \leq T)$ to time T.

An alternative Strike-separable Pricing Formula form evaluating $C(T, R), for R > 0$, is given by

Proposition 5.2.2.

$$C(T,R) = \phi_{X_m}(-i) - \frac{\sqrt{R}}{\pi}I(R)$$

where

$$I(R) = \int_0^1 \cos(\frac{1-t}{t}\log R)s_1(t)dt + \int_0^1 \sin(\frac{1-t}{t}\log R)s_2(t)dt$$
$$s_1(t) = \frac{\Re\big[\phi_{X_m}(-\frac{1-t}{t} - \frac{i}{2})\big]}{1 - 2t + 5/4t^2} \text{ and } s_2(t) = \frac{\Im\big[\phi_{X_m}(-\frac{1-t}{t} - \frac{i}{2})\big]}{1 - 2t + 5/4t^2}$$

The Characteristic Function of the Asset Price Maximum $\phi_{X_m}(z)$ calculated by a recursive relationship between the characteristic function of the maximum of a process over m monitoring points and the characteristic function(s) over the previous m–1 monitoring points as

$$\phi_{X_0}(z) = 1, \phi_{X_m}(z) = \frac{1}{m}\sum_{k=0}^{m-1}\phi_{X_k}(z)a_{m-k}(z), for;\ m = 1, 2, ..$$

where $a_{m-k}(z) = E_Q(e^{izL_{j^*+m-k}})$. The Expansion Formula for the recursion is given by

$$\phi_{X_m}(z) = Coeff_{t^m}\Big(\sum_{j=0}^{m}\frac{1}{j!}\Big(\sum_{k=1}^{m}\frac{a_k(z)}{k}t^k\Big)\Big)$$

where $Coeff_{t^m}$ is the coefficient in front of t^m , for $m = 1, 2, ..$

5.3 Proposed New Method

In this section we present two main theorems related to Look-back options. Let $M(t) = \max_{0\leq u\leq t} S(u)$, $0 \leq t \leq T$. Then M is an increasing continous process and thus has no Brownian component. There are four different types of Look-back options according to the payoff function. Let T be the time of expiry of the option. For *fixed strike* (E) call and put Look-back options payoffs are given by $(M(T) - E)^+$ and $(E - M(T))^+$ respectively. For *floating strike* call and put Look-back options the payoffs are given by $(S(T) - M(T))^+$ and $(M(T) - S(T))^+$ respectively. In this section we develop a technique for pricing floating strike put Look-back options. Option pricing for other Look-back options can be done with very similar procedure. We first show that the price of the Look-back option is given by a PIDE.

We assume that the stock price defined in 2.5.1 as exponential Lévy process with $Z(t)$ having finite moments, i.e $\int_{|x|\geq 1} |x|^p \nu(dx) < \infty$ for all positive integer p (see [68]) and also contains jumps of finite variation, i.e $\int_{|x|\leq 1} |x|\nu(dx) < \infty$. The examples of such a class of Lévy processes are Compound Poisson, VG and CGMY processes for $0 < Y < 1$.

Theorem 5.3.1. *The price of floating put Look-back option $P(t, S(t), M(t))$, where the stock-price dynamics is given by* (2.5.1)*, is given by*

$$\begin{aligned}
&\frac{\partial P}{\partial t} + rS\frac{\partial P}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} - rP \\
&+ \left[\int_{-\infty}^{\ln\frac{M}{S}} P(t, Se^x, M)\nu_Q(dx) - \int_{-\infty}^{\infty} \Big\{P(t, S, M) + S(e^x - 1)\frac{\partial P}{\partial S}\Big\}\nu_Q(dx)\right] = 0, \quad (5.3.1)
\end{aligned}$$

with final condition

$$P(T, S, M) = M - S \text{ when } S < M \quad (5.3.2)$$

$$\frac{\partial P(t, S, M)}{\partial M} = 0 \text{ when } S = M \quad (5.3.3)$$

Proof. Let us assume there exists a smooth continuous function for put price given by $P : [0, T] \times [0, \infty] \times [0, \infty] \to \mathbb{R}$, $P \in \mathcal{C}^{1,2,2}$. Under an equivalent martingale measure $\mathbb{Q}$, the put price of floating type Look-back option can be written as

$$P(t, S(t), M(t)) = e^{-r(T-t)} \mathrm{E}_{\mathcal{Q}}\Big[\big(M(T) - S(T)\big)^+ |\mathcal{F}_t\Big]$$

where

$$M(T) = \max_{0\leq u\leq T} S(u) \ , \ 0 \leq u \leq T$$

From the dynamics of the stock price defined at (2.5.5), we have

$$\begin{aligned}
dS(t) &= S(t-)\Big\{rdt + \sigma dW_Q(t) + \int_R (e^x - 1)\widetilde{N_Q}(dt, dx)\Big\} \\
&= dR(t)(\text{Riemann part}) + dI(t) \text{ (Itô part) } + \Delta S \text{ (Jump part)}
\end{aligned}$$

The continuous part of S(t) is defined to be

$$dS^c(t) = rS(t)dt + \sigma S(t)dW_Q(t)$$

and jump part of S(t) is $\Delta S = S_t - S_t-$.

The process $M(t)$ is nonnegative and increasing but not strictly increasing because it will have "Flat Spot" when $S(t)$ drops below its maximum (when $S(t) < M(t)$) and the payoff of the contact has nonzero returns only these points. Interestingly, $M(t)$ is actually constant at those points and therefore, it has neither I$t\hat{o}$ part nor Riemann part and can't have jump part also when payoff is nonzero. Therefore $\Delta M = 0$ or $M_t = M_t-$ and $dM = 0$ when $S(t) < M(t)$ otherwise positive.

Since all Levy processes (square-integratble) (2.3.14) are semimartingales, we can apply Itô formula for semimartingale on $\tilde{P}$. Let $\widetilde{P}(t, S(t), M(t)) = e^{r(T-t)}P(t, S(t), M(t))$ and using two-dimensional I$t\hat{o}$ formula for L$\acute{e}$vy Process, we can write

$$\begin{aligned}
d\widetilde{P}(t, S(t), M(t)) = e^{r(T-t)} \Bigg[& \frac{\partial P(t,S,M)}{\partial S} dS + \frac{\partial P(t,S,M)}{\partial M} dM + \frac{\partial P(t,S,M)}{\partial t} dt \\
& - rP(t,S,M) + \frac{1}{2}\frac{\partial^2 P(t,S,M)}{\partial S^2} d[S^c, S^c](t) \\
& + \frac{1}{2}\frac{\partial^2 P(t,S,M)}{\partial M^2} d[M^c, M^c](t) + \frac{\partial^2 P(t,S,M)}{\partial S \partial M} d[S^c, M^c](t) \\
& + P(t, S_t, M_t) - P(t, S_{t^-}, M_{t^-}) - \frac{\partial P(t,S,M)}{\partial S}\Delta S - \frac{\partial P(t,S,M)}{\partial M}\Delta M \Bigg]
\end{aligned}$$

Now we know that

$$d[S^c, S^c](t) = \sigma^2 S^2 d[W_Q(t), W_Q(t)](t) = \sigma^2 S^2 dt$$

and there is no I$t\hat{o}$ part of maximum process M, therefore we can write the Quadratic variations $d[M^c, M^c](t) = 0$ and $d[S^c, M^c](t) = 0$.
Let us consider $S(t) = S$, $M(t) = M$ and substituting the quadratic variations into

above gives,

$$
\begin{aligned}
&d\widetilde{P}(t,S(t),M(t)) = \\
&e^{r(T-t)}\Big[\frac{\partial P}{\partial t}dt - rPdt + \frac{\partial P}{\partial M}dM\Big] \\
&+\frac{\partial P}{\partial S}\left(rSdt + \sigma S(t)dW_Q(t) + S(t)\int_{\mathbb{R}}(e^x-1)\widetilde{N_Q}(dt,dx)\right) \\
&+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2}dt + P(t,S_{t^-}+\Delta S, M_{t^-}) - P(t,S_{t^-},M_{t^-}) - \frac{\partial P}{\partial S}\Delta S\Big] \\
&= e^{r(T-t)}\left[\frac{\partial P}{\partial t} + rS\frac{\partial P}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2} - rP\right. \\
&+\left.\left(\int_{-\infty}^{\ln\frac{M}{S}} P(t,Se^x,M)\nu_Q(dx) - \int_{-\infty}^{\infty}\Big\{P(t,S,M) + S(e^x-1)\frac{\partial P}{\partial S}\Big\}\nu_Q(dx)\right)\right]dt \\
&+e^{r(T-t)}\sigma S(t)\frac{\partial P}{\partial S}dW_Q(t) \\
&+e^{r(T-t)}\Big\{\int_{-\infty}^{\ln\frac{M}{S}} P(t,Se^x,M)\widetilde{N_Q}(dt,dx) - \int_{-\infty}^{\infty}P(t,S,M)\widetilde{N_Q}(dt,dx)\Big\} \\
&+e^{r(T-t)}\frac{\partial P}{\partial M}dM \\
&= a(t)dt + dM'(t) + e^{r(T-t)}\frac{\partial P}{\partial M}dM
\end{aligned}
$$

where

$$
\begin{aligned}
a(t) = e^{r(T-t)}&\left[\frac{\partial P}{\partial t} + rS\frac{\partial P}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2}rP - rP\right. \\
&+\left.\int_{-\infty}^{\ln\frac{M}{S}} P(t,Se^x,M)\nu_Q(dx) - \int_{-\infty}^{\infty}\Big\{P(t,S,M) + S(e^x-1)\frac{\partial P}{\partial S}\Big\}\nu_Q(dx)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&dM'(t) = e^{r(T-t)}\sigma S(t)\frac{\partial P}{\partial S}dW_Q(t)e^{r(T-t)}\Big\{\int_{-\infty}^{\ln\frac{M}{S}} P(t,Se^x,M)\widetilde{N_Q}(dt,dx) \\
&-\int_{-\infty}^{\infty}P(t,S,M)\widetilde{N_Q}(dt,dx)\Big\}
\end{aligned}
$$

The first term in $M'(t)$ is a martingale and second term is also a martingale by proposition 2.16 from Cont and Peter (2004).So we conclude that $M'(t)$ is a martingale.Therefore, $\widetilde{P}(t,S(t),M(t)) = \mathrm{E}_Q[H(S(t))|\mathcal{F}_t]$ and $M'(t)$ both are martingales, then $\widetilde{P}(t,S(t),M(t)) - M'(t)$ is also a martingale.But $\widetilde{P}(t,S(t),M(t)) - M'(t) = \int_0^t a(s)ds$ is a continuous process with finite variation, so, by [[58],Theorem 4.13-4.50] we must have $a(t) = 0$ almost surely.The dM(t) term is naturally zero on the "flat spots" of M(t) (i.e., when $S(t) < M(t)$). However, at the times when M(t) increases, which are the times when S(t) = M(t), the term $e^{r(T-t)}\frac{\partial P}{\partial M}$ must be zero because dM(t) is "positive" which gives us the boundary condition.

Therefore, we have a partial integro-differential equation(PIDE) for Lookback option is given by,

$$\frac{\partial P}{\partial t} + rS\frac{\partial P}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} - rP$$
$$+ \left[\int_{-\infty}^{\ln\frac{M}{S}} P(t, Se^x, M)\nu_Q(dx) - \int_{-\infty}^{\infty}\Big\{P(t,S,M) + S(e^x - 1)\frac{\partial P}{\partial S}\Big\}\nu_Q(dx)\right] = 0$$

for $0 < t \leq T$ and $0 < S < M < \infty$ and the boundary conditions

$$P(T, S, M) = M - S \text{ when } S < M$$
$$\frac{\partial P(t,S,M)}{\partial M} = 0 \text{ when } S = M$$

□

Theorem 5.3.2. *The closed-form Fourier Pricing expression of Look-back put option of floating type $P(t, S(t), M(t))$ when stock price is driven by Lévy process under assumption 2.5.1 and $\alpha > 0$, is given by*

$$P(t, S(t), M(t)) = Mg(t, \log(\frac{M}{S})) \tag{5.3.4}$$

where

$$g(t,k) = \frac{e^{\alpha k}}{\sqrt{2\pi}}\Big[-\int_{-\infty}^{0}\frac{1}{1+2\alpha}\big\{e^{\alpha\zeta} - e^{-(1+\alpha)\zeta}\big\}\Phi(\zeta - k)d\zeta + \int_{0}^{\infty}\big\{e^{-\alpha\zeta} - e^{-(1+\alpha)\zeta}\big\}\Phi(\zeta - k)d\zeta\Big]$$

and

$$\Phi(\zeta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\eta(\zeta)}e^{\psi(\eta)(T-t)}d\eta$$

with

$$\psi(\eta) \;=\; -(1+\alpha)\Big\{r - \frac{1}{2}\sigma^2\alpha\Big\} + \int_{-1}^{1}(e^{-\alpha x} - 1)\nu_Q(x)dx - \frac{1}{2}\eta^2\Big\{\sigma^2 + \int_{-1}^{1} e^{-\alpha x}x^2\nu_Q(x)dx\Big\}$$

and α must satisfy

$$r - \frac{1}{2}(2\alpha+1)\sigma^2 - \int_{-\infty}^{\infty}(e^x - 1)\nu_Q(x)dx + \int_{-1}^{1} e^{-\alpha x}x\nu_Q(x)dx = 0 \tag{5.3.5}$$

Proof. Let us consider

$$f(t,z) \;=\; P\big(t,\frac{S}{M},1\big),\;\; 0\le t\le T,\;\; \text{considering } z=\frac{S}{M}$$

where $f(t,z)$ is continuous and differentiable everywhere according to the definition of put price.
Then,

$$P(t,S,M)=Mf\big(t,\frac{S}{M}\big);\, P(t,cS,M)=Mf\big(t,c\frac{S}{M}\big);\, \frac{\partial P(t,S,M)}{\partial t}=M\frac{\partial f\big(t,\frac{S}{M}\big)}{\partial t};$$
$$\frac{\partial P(t,S,M)}{\partial S}=\frac{\partial f\left(t,\frac{S}{M}\right)}{\partial z};\,\frac{\partial^2 P(t,S,M)}{\partial S^2}=\frac{1}{M}\frac{\partial^2 f\left(t,\frac{S}{M}\right)}{\partial z^2}$$

Substituting into above, we have another PIDE as

$$\begin{aligned}\frac{\partial f(t,z)}{\partial t} &+ rz\frac{\partial f(t,z)}{\partial z}+\frac{1}{2}\sigma^2 z^2\frac{\partial^2 f(t,z)}{\partial z^2}-rf(t,z)\\ &+\left[\int_{-\infty}^{-\ln z} f(t,e^x z)\nu_Q(dx)-\int_{-\infty}^{\infty}\Big\{f(t,z)+z\frac{\partial f(t,z)}{\partial z}(e^x-1)\Big\}\nu_Q(dx)\right]=0\end{aligned}$$

with boundary conditions

$$\begin{aligned} f(T,z) &= (1-z) \text{ when } 0<z\le 1\\ f(t,z) &= \frac{\partial f(t,z)}{\partial z} \text{ when } z=1\end{aligned}$$

Let us consider $z=e^{-k}$, then $f(t,e^{-k})=g(t,k)$, then we can rewrite as

$$\frac{\partial g(t,k)}{\partial t}+\mathcal{L}g(t,k)=0$$

$$\begin{aligned}\mathcal{L}g(t,k) &= -(r-\frac{1}{2}\sigma^2)\frac{\partial g(t,k)}{\partial k}+\frac{1}{2}\sigma^2\frac{\partial^2 g(t,k)}{\partial k^2}-rg(t,k)\\ &+\left[\int_{-\infty}^{k} g(t,k-x)\nu_Q(dx)-\int_{-\infty}^{\infty}\Big\{g(t,k)-\frac{\partial g(t,k)}{\partial k}(e^x-1)\Big\}\nu_Q(dx)\right]\end{aligned}$$

with boundary and initial conditions

$$\begin{aligned} g(T,k) &= (1-e^{-k}) \text{ when } 0\le k<\infty\\ \text{and} &\quad \left[g(t,k)+\frac{\partial g(t,k)}{\partial k}\right]\Big|_{k=0}=0\end{aligned}$$

The function $g(t,k)$ is a smooth, continuous and bounded to e^{-rt} as $k\to\infty$ (or $S\to 0$). Let us consider a function $\tilde{g}(t,k)=g(t,k)e^{-\alpha k}$ which vanishes to zero when $k\to\infty$ and for $\alpha>0$. Then, the new PIDE is

$$\frac{\partial \tilde{g}(t,k)}{\partial t} + \mathcal{L}\tilde{g}(t,k) = 0 \tag{5.3.6}$$

where

$$\begin{aligned}
\tilde{\mathcal{L}}\tilde{g}(t,k) &= -(\alpha+1)\Big\{r - \frac{1}{2}\sigma^2\alpha\Big\}\tilde{g}(t,k) \\
&- \Big\{r - \frac{1}{2}\sigma^2(2\alpha+1)\Big\}\frac{\partial \tilde{g}(t,k)}{\partial k} + \frac{1}{2}\sigma^2\frac{\partial^2 \tilde{g}(t,k)}{\partial k^2} \\
&+ \int_{-\infty}^{k} \tilde{g}(t,k-x)e^{-\alpha x}\nu_Q(x)dx - \int_{-\infty}^{\infty}\Big\{\tilde{g}(t,k) - (e^x-1)\frac{\partial \tilde{g}(t,k)}{\partial k}\Big\}\nu_Q(x)dx
\end{aligned}$$

with initial and boundary condition

$$\begin{aligned}
I.C &: \Big[(1+\alpha)\tilde{g}(t,k) + \frac{\partial \tilde{g}(t,k)}{\partial k}\Big]\Big|_{k=0} = 0 \\
B.C &: \tilde{g}(T,k) = e^{-\alpha k}(1-e^{-k}), \text{ when } 0 \le k < \infty
\end{aligned}$$

Let us consider the Cauchy problem

$$\frac{\partial \tilde{g}_*(t,k)}{\partial t} + \tilde{\mathcal{L}}\tilde{g}_*(t,k) = 0, \tag{5.3.7}$$

with

$$\begin{aligned}
\tilde{g}_*(T,k) &= e^{-\alpha k}(1-e^{-k}), \text{when } 0 \le k < \infty \\
&= \Psi(k) \text{ say, when } -\infty < k < 0
\end{aligned} \tag{5.3.8}$$

Applying Fourier transform of Cauchy problem (5.3.7) for solving and using composition theorem, we obtain

$$\begin{aligned}
\tilde{g}_*(t,k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\eta k}\hat{g}_*(T,\eta)e^{\psi(\eta)(T-t)}d\eta \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{g}_*(T,\zeta)\Phi(\zeta-k)d\zeta \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \tilde{g}_*(T,\zeta)\Phi(\zeta-k)d\zeta + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \tilde{g}_*(T,\zeta)\Phi(\zeta-k)d\zeta
\end{aligned} \tag{5.3.9}$$

where $\Phi(\zeta) = \mathcal{F}_\zeta\Big[e^{\psi(\eta)(T-t)}\Big] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\eta(\zeta)}e^{\psi(\eta)(T-t)}d\eta$ and

$$\begin{aligned}
\psi(\eta) &= -(1+\alpha)\Big\{r - \frac{1}{2}\sigma^2\alpha\Big\} - i\eta\Big\{r - \frac{1}{2}(2\alpha+1)\sigma^2 - \int_{-\infty}^{\infty}(e^x-1)\nu_Q(x)dx\Big\} \\
&- \frac{1}{2}\sigma^2\eta^2 + \int_{-\infty}^{\infty}\Big\{e^{-i\eta x}e^{-\alpha x} - 1\Big\}\nu_Q(x)dx.
\end{aligned} \tag{5.3.10}$$

Since

$$\int_{-\infty}^{\infty}\Big\{e^{-i\eta x}e^{-\alpha x}-1\Big\}\nu_Q(x)dx=\int_{-\infty}^{\infty}\Big\{e^{-\alpha x}\cos\eta x-1\Big\}\nu_Q(x)dx-i\int_{-\infty}^{\infty}e^{-\alpha x}\sin\eta x\;\nu_Q(x)dx$$

and we have used finite activity Lévy process where measure ν_Q is concentrated in $[-1,1]$, the following approximation can be assumed

$$\int_{-\infty}^{\infty}\Big\{e^{-\alpha x}\cos\eta x-1\Big\}\;\nu_Q(x)dx\approx\int_{-1}^{1}\Big\{e^{-\alpha x}-1-\frac{\eta^2}{2}x^2e^{-\alpha x}\Big\}\;\nu_Q(x)dx$$

And (5.3.11)

$$\int_{-\infty}^{\infty}e^{-\alpha x}\sin\eta x\;\nu_Q(x)dx\approx\eta\int_{-1}^{1}e^{-\alpha x}\;x\nu_Q(x)dx$$

Therefore,

$$\begin{aligned}&\psi(\eta)\\ =\;&-(1+\alpha)\Big\{r-\frac{1}{2}\sigma^2\alpha\Big\}+\int_{-1}^{1}(e^{-\alpha x}-1)\nu_Q(x)dx-\frac{1}{2}\eta^2\Big\{\sigma^2+\int_{-1}^{1}e^{-\alpha x}x^2\nu_Q(x)dx\Big\}\\ -\;&i\eta\Big\{r-\frac{1}{2}(2\alpha+1)\sigma^2-\int_{-\infty}^{\infty}(e^x-1)\nu_Q(x)dx+\int_{-1}^{1}e^{-\alpha x}x\nu_Q(x)dx\Big\}\end{aligned}$$

Let us consider $\alpha=\alpha^*$ such that

$$r-\frac{1}{2}(2\alpha^*+1)\sigma^2-\int_{-\infty}^{\infty}(e^x-1)\nu_Q(x)dx+\int_{-1}^{1}e^{-\alpha^*x}x\nu_Q(x)dx=0 \qquad (5.3.12)$$

and the value of α^* can be found numerically in general. The function $\psi(\eta)$ now becomes symmetric (otherwise $\psi(\eta)=\psi(-\eta)$) and also $\Phi(\zeta)=\Phi(-\zeta)$. Our next step would be to find the unknown function Ψ such that it satisfies I.C of (5.3.6) for all t so that I.C can be discarded. We can now write

$$\begin{aligned}&\int_{-\infty}^{\infty}\Big\{(1+\alpha)\tilde{g}_*(T,\zeta)+\frac{\partial\tilde{g}_*(T,\zeta)}{\partial\zeta}\Big\}\Phi(\zeta)d\zeta=0\\ \Longrightarrow\;&\int_{0}^{\infty}e^{-\alpha\zeta}\Phi(\zeta)d\zeta+\int_{-\infty}^{0}\big[(1+\alpha)\Psi(\zeta)+\Psi_\zeta(\zeta)\big]\Phi(\zeta)d\zeta=0 \qquad (5.3.13)\end{aligned}$$

Therefore, one of the possibility is $(1+\alpha)\Psi(\zeta)+\Psi_\zeta(\zeta)+e^{\alpha\zeta}=0$ with $\Psi(0)=0$. In our case we can choose any function $\Psi(\zeta)$ and that does not impact the solution of (5.3.6) since we are only interested only when $k<0$. Therefore, solving we have

$$\Psi(\zeta)=-\frac{1}{1+2\alpha}\Big[e^{\alpha\zeta}-e^{-(1+\alpha)\zeta}\Big] \qquad (5.3.14)$$

Finally, we can write

$$\begin{aligned}\tilde{g}_*(t,k)\;&=\;-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\frac{1}{1+2\alpha}\Big[e^{\alpha\zeta}-e^{-(1+\alpha)\zeta}\Big]\Phi(\zeta-k)d\zeta\\ &+\;\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\Big[e^{-\alpha\zeta}-e^{-(1+\alpha)\zeta}\Big]\Phi(\zeta-k)d\zeta\end{aligned}$$

□

We conclude this section with the computation of *sensitivities* for floating type put Look-back options as follows

Theorem 5.3.3. *The pricing* sensitivities *of the Look-back floating type put option are given by*

$$\Delta(t,S,M) = -\frac{M}{S}\frac{e^{\alpha \ln \frac{M}{S}}}{\sqrt{2\pi}} \times$$

$$\Big[-\int_{-\infty}^{0} \frac{1}{1+2\alpha}\big\{e^{\alpha\zeta} - e^{-(1+\alpha)\zeta}\big\}\Phi_k(\zeta - \ln\frac{M}{S})d\zeta + \int_{0}^{\infty}\big\{e^{-\alpha\zeta} - e^{-(1+\alpha)\zeta}\big\}\Phi_k(\zeta - \ln\frac{M}{S})d\zeta\Big]$$

$$\Gamma(t,S,M) = \frac{M^2}{S}\frac{e^{\alpha \ln \frac{M}{S}}}{\sqrt{2\pi}} \times$$

$$\Big[-\int_{-\infty}^{0} \frac{1}{1+2\alpha}\big\{e^{\alpha\zeta} - e^{-(1+\alpha)\zeta}\big\}\Phi_k(\zeta - \ln\frac{M}{S})d\zeta + \int_{0}^{\infty}\big\{e^{-\alpha\zeta} - e^{-(1+\alpha)\zeta}\big\}\Phi_k(\zeta - \ln\frac{M}{S})d\zeta$$

$$-\int_{-\infty}^{0} \frac{1}{1+2\alpha}\big\{e^{\alpha\zeta} - e^{-(1+\alpha)\zeta}\big\}\Phi_{kk}(\zeta - \ln\frac{M}{S})d\zeta + \int_{0}^{\infty}\big\{e^{-\alpha\zeta} - e^{-(1+\alpha)\zeta}\big\}\Phi_{kk}(\zeta - \ln\frac{M}{S})d\zeta\Big]$$

$$\Theta(t,S,M) = M\frac{e^{\alpha \ln \frac{M}{S}}}{\sqrt{2\pi}} \times$$

$$\Big[-\int_{-\infty}^{0} \frac{1}{1+2\alpha}\big\{e^{\alpha\zeta} - e^{-(1+\alpha)\zeta}\big\}\Phi_t(\zeta - \ln\frac{M}{S})d\zeta + \int_{0}^{\infty}\big\{e^{-\alpha\zeta} - e^{-(1+\alpha)\zeta}\big\}\Phi_t(\zeta - \ln\frac{M}{S})d\zeta\Big]$$

where

$$\begin{aligned}
\Phi(\zeta) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\eta\zeta}e^{\psi(\eta)(T-t)}d\eta \\
\Phi_k(\zeta) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\eta\zeta}\big(i\eta+\alpha\big)e^{\psi(\eta)(T-t)}d\eta \\
\Phi_{kk}(\zeta) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\eta\zeta}\big(i\eta+\alpha\big)^2e^{\psi(\eta)(T-t)}d\eta \\
\Phi_t(\zeta) &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\eta\zeta}\psi(\eta)e^{\psi(\eta)(T-t)}d\eta
\end{aligned}$$

Proof. The *sensitivities* are given by

$$\Delta(t,S,M) = \frac{\partial P}{\partial S}; \quad \Gamma(t,S,M) = \frac{\partial^2 P}{\partial S^2}; \quad \Theta(t,S,M) = \frac{\partial P}{\partial t}.$$

Hence the results follow from various differentiations of (5.3.4). □

5.4 Calibration of martingale measure $\mathbb{Q}$

In the last section, it is concluded that there exists multiple equivalent martingale measures $\mathbb{Q}$ depending on the choices of functions $H(t,x)$. This is leading to incompleteness while modeling with Lévy process. An ad-hoc choice (for example- Esscher transform) may not give option prices consistent with market prices of options. Therefore, we intend to estimate the function $H(t,x)$ by calibration of put price available in the market with the pricing model using optimal control.

The pricing expression derived in Theorem 5.3.2 uses a modified Lévy measure ν_Q, where $\nu_Q(dx)dt = e^{H(t,x)}\nu(dx)dt$. Under the assumption that ν_Q is not changing over time, we can write (2.5.3) as,

$$\nu_Q(dx) = e^{H(x)}\nu(dx) = \sum_{n\geq 0} h_n \frac{x^n}{n!}\nu(dx). \tag{5.4.1}$$

The function $e^{H(x)}$ is approximated by the Taylor series and is defined by coefficients h_n, $n = 0, 1, 2, \ldots$. Our objective is to estimate both functions $e^{H(x)}$ and $\nu(x)$.

We consider the (observed) market prices $P^*(T_i, S_i, M_i)$, $i = 1, \ldots, n$, for a set of liquid put options. The objective is to find constants ν such that

$$P^\nu(T_i, S_i, M_i) = P^*(T_i, S_i, M_i),$$

where P^ν is the option price computed for parameters ν. The popular approach to non-linear least squares is

$$(\nu^*) = \arg\inf_{\nu} \sum_{i=1}^{N} \{P^\nu(T_i, S_i, M_i) - P^*(T_i, S_i, M_i)\}^2.$$

The usual formulations of the inverse problems via nonlinear least squares are ill-posed and in [16] a regularization method is proposed on relative entropy. In [16], the calibration problem is reformulated into problem of finding a risk-neutral jump-diffusion model that reproduces the observed option prices and has the smallest possible relative entropy with respect to a chosen prior model. Our approach of calibration is based on the optimal control technique with regularization for estimation which avoids ill-posed issues.

5.4.1 The optimal control problem

We define a controlled dynamical system by a differential equation with control parameters $u_1(t), u_2(t), \ldots u_n(t)$, as

$$\varphi_t = f(t, \varphi, u_1, u_2, \ldots, u_n), \tag{5.4.2}$$
$$\varphi(0,k) = \varphi_0, \tag{5.4.3}$$

where $u_j : [0,T] \to \mathbb{R}$ is a function with a given control, a state function $\varphi : \mathcal{C}^{1,2}([0,T]\times\mathbb{R}) \to \mathbb{R}$, and $f : [0,T] \times \mathcal{C}^{1,2}([0,T]\times\mathbb{R}) \times \mathbb{R} \times \ldots \times \mathbb{R} \to \mathbb{R}$. For a given payoff function $r : [0,T] \times \mathcal{C}^{1,2}([0,T]\times\mathbb{R}) \times \mathbb{R} \times \ldots \times \mathbb{R} \to \mathbb{R}$, the optimal control problem is defined to find choice of control $u_1, u_2, .., u_n$ that minimizes the following objective function

$$J(u_1, u_2, \ldots, u_n, \varphi_0) = \inf_{u_1, u_2, \ldots, u_n} \int_0^T r(t, \varphi, u_1, u_2, \ldots, u_n)dt. \tag{5.4.4}$$

Following the Pontryagin minimum principle, we introduce a Lagrange multiplier function $\lambda(t)$. Then, we have

$$\begin{aligned}\mathcal{C}(\varphi, u_1, u_2, ..., u_n, \lambda) &= \int_0^T \Big[r(t, \varphi, u_1, u_2, ..., u_n) - \lambda(t)\Big(f(t, \varphi, u_1, u_2, ..., u_n) - \varphi_t\Big)\Big] dt \\ &= \int_0^T \Big[-\mathcal{H}(t, \varphi, u_1, u_2, ..., u_n, \lambda) + \lambda(t)\varphi_t\Big] dt,\end{aligned}$$

where the Hamiltonian $-\mathcal{H}(t, \varphi, u_1, u_2, ..., u_n, \lambda) = r(t, \varphi, u_1, u_2, ..., u_n) - \lambda f(t, \varphi, u_1, u_2, ..., u_n)$. The solution of (5.4.4) is found by extremizing $\mathcal{C}$, which is either maximizing H or minimizing $-H$. Differentiating with respect to φ,$u_1, u_2, ..., u_n$ and λ, we have

$$\begin{aligned}\delta\mathcal{C} &= \int_0^T \Big[-\mathcal{H}_\varphi \delta\varphi - (\mathcal{H}_{u_1}\delta u_1 + .. + \mathcal{H}_{u_n}\delta u_n) + \Big(-\mathcal{H}_\lambda + \varphi_t\Big)\delta\lambda(t) + \lambda(t)\delta\varphi_t\Big] dt + \\ &\quad \lambda(T)\delta\varphi(T, k) \\ &= \int_0^T \Big[(-\mathcal{H}_\varphi - \lambda_t)\delta\varphi - (\mathcal{H}_{u_1}\delta u_1 + .. + \mathcal{H}_{u_n}\delta u_n) + \Big(-\mathcal{H}_\lambda + \varphi_t\Big)\delta\lambda(t)\Big] dt + \lambda(T)\delta\varphi(T, k).\end{aligned}$$

Consequently, $\delta\mathcal{C} = 0$ gives

$$\begin{gathered}\mathcal{H}_{u_1}(t, \varphi, u_1, \lambda), ...\mathcal{H}_{u_n}(t, \varphi, u_n, \lambda) = 0 \\ \varphi_t = \mathcal{H}_\lambda(t, \varphi, u_1, u_2, ..., u_n, \lambda) \\ \lambda_t = -\mathcal{H}_\varphi(t, \varphi, u_1, u_2, ..., u_n, \lambda);\ \lambda(T) = 0,\end{gathered}$$

where $\mathcal{H}_{u_j}$, $\mathcal{H}_\lambda$ and $\mathcal{H}_\varphi$ denote the Gateaux-derivatives of $\mathcal{H}$ w.r.t. u_j, λ and φ respectively. Note that the Gateaux-derivatives of a functional $A(f) : \mathcal{B} \to \mathbb{R}$ is defined as

$$D_h A(f) = \lim_{\epsilon \to 0} \frac{A(f + \epsilon\xi) - A(f)}{\epsilon} = \frac{d}{d\epsilon} A(f + \epsilon\xi)|_{\epsilon=0},$$

where Banach space $\mathcal{B}$ is in $\mathbb{R}^n$.

5.4.2 Our approach

We assume $k(t) = \log(\frac{M(t)}{S(t)}) = k$. In our case, the parameters of the Lévy measures ν_Q represented by (5.4.1) are $h_1, h_2, ...h_n \in \mathbb{R}$ for $H(x)$, and $u_1, u_2, ..., u_n \in \mathbb{R}$ for ν are considered as the control. For example the parameters $u_1 = \sigma', u_2 = \nu', u_3 = \theta'$ for VG and $u_1 = C, u_2 = G, u_3 = M', u_3 = Y$ for $CGMY$ process. We find these control parameters by minimizing the error between the model price $P(t, S, M)$ in (5.3.4) and market price of the put contract $P^{Market}(t, S, M)$. This is equivalent to minimization of the error between $\tilde{g}(t, k; h_1, .., h_n, u_1, .., u_n)$ derived from model and fixed values of $g_m(t, k) = \frac{P^{\text{Market}}(t,S,M)}{M}$, computed from available market price. Therefore, the objective function with regularization is defined as follows.

$$\inf_{h_1,..,h_n,u_1,..,u_n} \int_0^T \Big[\Big\{\tilde{g}(t, k; h_1, .., h_n, u_1, .., u_n) - \tilde{g}_m(t, k)\Big\}^2\Big] dt, \qquad (5.4.5)$$

subject to

$$\frac{\partial\tilde{g}(t,k)}{\partial t} \quad = \quad \tilde{\mathcal{L}}\tilde{g}(t,k;h_1,..,h_n,u_1,..,u_n)$$

with boundary condition

$$\tilde{g}(T,k) = \begin{cases} \Big[e^{-\alpha k} - e^{-(1+\alpha)k}\Big], \text{ if } 0 \le k < \infty, \\ -\frac{1}{1+2\alpha}\Big[e^{\alpha k} - e^{-(1+\alpha)k}\Big], \text{ if } -\infty < k < 0. \end{cases}$$

Let us consider the functional with Lagrange multiplier $\lambda(t)$

$$\begin{aligned} & I\big(\tilde{g}(t,.),h_1,..,h_n,u_1,...,u_j\big) \\ &= \int_0^T \Big[\big\{\tilde{g}(t,k;h_1,..,h_n,u_1,..,u_n) - \tilde{g}_m(t,k)\big\}^2 + \lambda(t)\big\{\tilde{g}_t - \tilde{\mathcal{L}}\tilde{g}(t,k;h_1,..,h_n,u_1,..,u_n)\big\}\Big]dt \\ &= \int_0^T \Big[\mathcal{H}(t,k;h_1,..,h_n,u_1,..,u_n)\Big]dt + \lambda(T)\tilde{g}(T,k), \end{aligned}$$

where

$$\begin{aligned} & \mathcal{H}(t,k;h_1,..,h_n,u_1,..,u_n) = \\ & \Big[\tilde{g}(t,k;h_1,..,h_n,u_1,..,u_n) - \tilde{g}_m(t,k)\Big]^2 - \lambda(t)\tilde{\mathcal{L}}\tilde{g}(t,k;h_1,..,h_n,u_1,..,u_n) \\ & - \lambda_t\tilde{g}(t,k;h_1,..,h_n,u_1,..,u_n) \end{aligned}$$

We replace $\tilde{g}(t,.)$ by $\tilde{g}(t,.) + \epsilon\xi(t)$ and the variation $\delta I = \Big[\frac{d}{d\epsilon}I(\tilde{g}+\epsilon\xi)\Big]_{\epsilon=0} = 0$ for an extremum condition. This yields

$$\begin{aligned} & \lambda_t + \varrho\lambda - 2\big[\tilde{g}(t,k;h_1,..,h_n,u_1,..,u_n) - \tilde{g}_m(t,k)\big] = 0, \\ & \lambda(T) = 0, \end{aligned}$$

where

$$\varrho = -(\alpha+1)\Big\{r - \frac{1}{2}\sigma^2\alpha\Big\} + \int_{-\infty}^{k} e^{-\alpha x}\nu_Q(dx) - \int_{-\infty}^{\infty}\nu_Q(dx)$$

Consequently, we have

$$\lambda(t) = -2e^{-\varrho t}\int_t^T e^{\varrho y}\big\{\tilde{g}(y,k;h_1,..,h_n,u_1,..,u_n) - \tilde{g}_m(y,k)\big\}dy. \tag{5.4.6}$$

Similarly,

$$\tilde{g}_t - \tilde{\mathcal{L}}\tilde{g}(t,k;h_1,..,h_n,u_1,..,u_n) = 0. \tag{5.4.7}$$

Table 5.1: Parameters used for Lévy processes

Model			Parameters
VG	$\sigma' = 0.0336$	$\nu' = 0.0070$	$\theta' = 0.4700$

The solution of (5.4.7) with boundary condition is

$$\begin{aligned}\tilde{g}(t,k) &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\frac{1}{1+2\alpha}\Big[e^{\alpha\zeta}-e^{-(1+\alpha)\zeta}\Big]\Phi(\zeta-k)d\zeta \\ &+ \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\Big[e^{-\alpha\zeta}-e^{-(1+\alpha)\zeta}\Big]\Phi(\zeta-k)d\zeta\end{aligned}$$

Therefore, we are set to determine the control h_j^* and u_j^* numerically with the help of (5.4.6) and (5.4.7)

$$\mathcal{H}_{h_j^*,u_j^*}(t,\tilde{g}^*(t,k;h_j^*,u_j^*),h_j^*,u_j^*,\lambda^*(\tau,h_j^*,u_j^*)) = 0. \tag{5.4.8}$$

5.4.3 Numerical results

We implement a pricing algorithm (Algorithm 3) for computing look-back put option price both with Monte Carlo simulation and proposed closed-form formula described in Theorems 5.3.2 and 5.3.3 with VG process described in Appendix 2.3.1 as one of the instances of the Lévy class. The parameters of $VG(\sigma,\nu,\theta)$ are determined by calibrating the models with the market quoted Stoxx50E call prices on 16 June 2006 using the FFT algorithm proposed by [44]. This procedure yields the parameter as described in Table 5.2. We employ these values for calculating the put option prices and sensitivities both by Monte Carlo simulation and our proposed method.

We first simulate a large number of paths of stock price $S(t)$ that follows dynamics described in (2.5.5) where the jump part of the price is governed by VG process with estimated parameters for a specific time to maturity. For each of path the maximum process $M(t)$ of the stock price and payoff $M(t)-S(t)$ is calculated over the path. Then the put price is the mean of all calculated payoffs for each path after discounting by the factor $e^{-r(T-t)}$. This process is repeated multiple times in a batch and the mean put price and its variance are computed for each time to maturity. The result of computed price and its variation for specific time to maturity plotted in Figure 5.1. We consider these simulated prices as reference price for comparing and validating our proposed new closed-form pricing expression. Our pricing formula described in Theorems 5.3.2 and 5.3.3 has a Fourier inverse integral which is computed numerically with inverse FFT method. The integral inside pricing expression is computed by a scalable method described in Appendix A.1.

The computed price with our pricing expression is now plotted along with prices by Monte Carlo simulation in Figure 5.1. We observe that the average price of the simulation is close to the prices calculated by our expression and goodness-of-fit shows

Algorithm 3 Algorithms for computing put price by Monte carlo and proposed closed-form formula

Input: Assume market parameters ν for $VG(\sigma, \nu, \theta)$, initial stock price S, its maximum M, price volatility σ, risk free interest rate r and the time to maturity $T-t$.

Output: Put price at any future time after t, $P(t, S, M)$ by Pricing expression and compared accuracy by Monte carlo simulation. Computed the sensitivities for the options.

1: {**Step 1**}
1.1 Simulate Jump $J(t)$ that follows a class of Lévy process such as $VG(\sigma, \nu, \theta)$.
1.2 Simulate Weiner process $W(t)$ that follows standard normal distribution.
1.3 $dS(t) = S(t-)\{r\,dt + \sigma dW(t) + dJ(t)\}$, $S(t) = S(t-) + dS(t)$ and $M_t = \max(S_t)$
$Payoff = avg(M_t - S_t)$
1.4 Put Price $P^{MC}(t, S, M)$= $\exp(-rt)$ mean {Payoff}
1.5 Computer 1.1 to 1.4 for n times and compute mean, variance of put price $P^{MC}(t, S, M)$

2: {**Step 2**}
The integral part of the expression calculated using Clenshaw-Curtis rule while computing $g(t, k)$ from 5.3.4 with the help of Fourier inverse transform described in the Appendix and $P^{Model}(t, S, M) = Mg(t, \ln \frac{M}{S})$

3: {**Step 3**}
For *goodness of fit* of the calibration, we use the absolute percentage error (APE), the average absolute error (AAE), the average relative percentage error (ARPE) and the root-mean-square error (RMSE) given by the following formulas.

$$APE = \frac{1}{\text{mean option price}} \sum_{options} \frac{|P^{MC} - P^{Model}|}{\text{number of options}},$$

$$AAE = \sum_{options} \frac{|P^{MC} - P^{Model}|}{\text{number of options}},$$

$$ARPE = \frac{1}{\text{number option price}} \sum_{options} \frac{|P^{MC} - P^{Model}|}{\text{number of options}},$$

$$RMSE = \sqrt{\sum_{options} \frac{(P^{MC} - P^{Model})^2}{\text{number of options}}}.$$

4: {**Step 4**} Similarly, the expression of *Greeks* of the Look-back defined in 5.3.15 are computed using Clenshaw-Curtis rule and Fourier inverse transform described in the Appendix.

that the parameters APE, AAE, ARPE and RMSE in Table 5.2 are reasonably small. In order to validate and investigate in detail, the difference between prices are also computed by two methods for various parameter values of S,M and, the time to Maturity $(T-t)$. We notice that the differences or errors are reasonably small. Put price change over the time to maturity, stock price and its maximum with $VG(\sigma,\nu,\theta)$ are provided in Table 5.3.

The put price $P(t,S)$ plotted with respect to stock price S and time to maturity $T-t$ in 5.2 and noticed that it is dreasing with the increase of both the parameters, which is complying with the definition of put price. The sensitivities such as Delta, Gamma and Theta are also plotted. We have also computed and plotted the sensitivities in Figure 5.2.

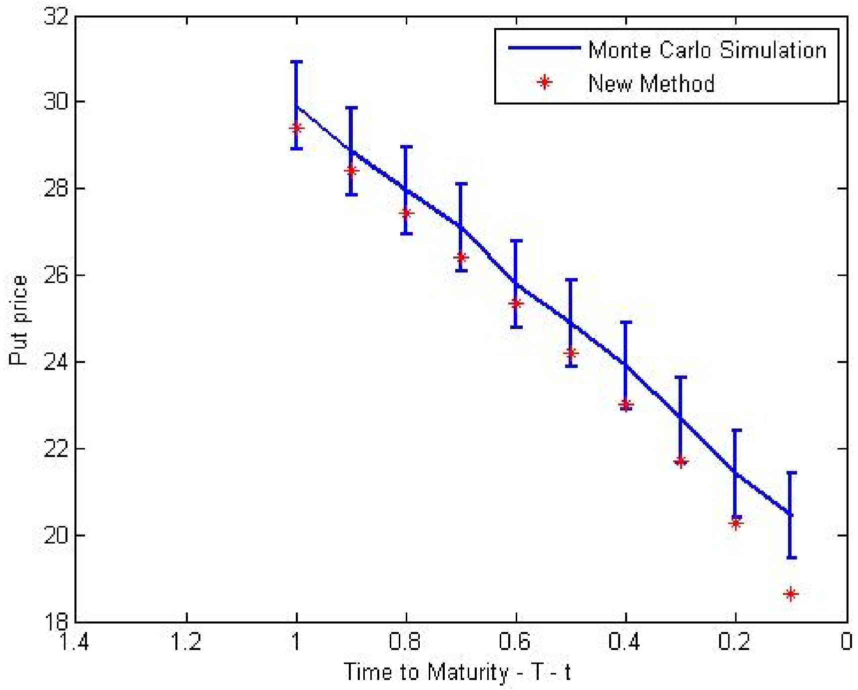

Figure 5.1: Comparison of Proposed Method vs Monte Carlo.

Table 5.2: Performance of above Lévy Model

Model	APE(%)	AAE	RMSE	ARPE(%)
VG	3.1283	0.7908	7.9082	0.2501

Table 5.3: Put price change over the time to maturity, stock price and its maximum with VG(σ, ν, θ)

$T-t$	S	M	Proposed Method Put	Monte Carlo Put	Difference(%) Put
1	100	120	29.4108	29.7395	1.1053
1	100	100	20.9627	22.8286	8.1735
0.8	80	100	24.4174	24.8012	1.5475
0.8	60	100	40.4758	38.5958	4.8710
0.4	80	100	21.1014	22.0571	4.3328
0.4	100	100	14.9697	14.4682	3.4662

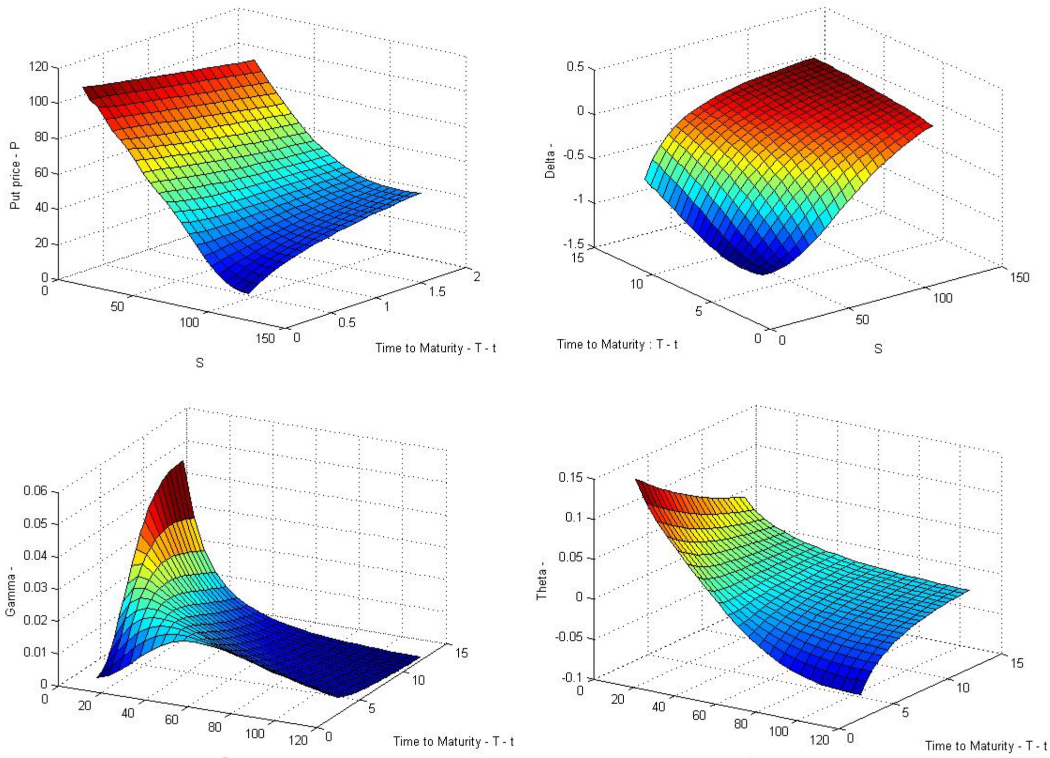

Figure 5.2: Put price and sensitivities of look-back put option with VG processes.

5.5 Conclusion

We have shown that it is possible to derive a PIDE and a closed-form arbitrage-free pricing formula for floating type look-back option under the exponential Lévy process. The formula is simple, easy to compute and can work for any class of Lévy process avoiding complicated numerical methods in the existing literature. The proposed method and the Monte Carlo simulation method are very close in terms of accuracy. We have also proposed a novel method of finding the risk neutral equivalent martingale measure by calibration from the available market prices. We hope that our method will be a satisfactory theoretical development for financial theorists and prove useful for the practitioners.

Chapter 6

Conclusion and Future research

6.1 Conclusion

Modeling stock price Our studies about the statistical properties of the stock price of major indices such as Google US, LLY and MSFT US Equity identifies the asymmetry and fat tails. The JB and KS tests confirms the non-normal distribution. When we try to fit a known class of Lévy process called NIG by Maximum Likelihood estimation, the result shows very close fit and this indicates the presence of jumps. Therefore, our approach of modeling stock price with exponential Lévy process seems appropriate to the real market movements.

Estimation of Risk Neutral Density The pricing models for the contracts with jump process such as Lévy suffers from a major challenge. This is due to fact that the Lévy market is incomplete and replication of contingent claim is not possible and risk neutral measure Q is not unique and unknown. The popular approaches such as Esscher transform, Relative Entropy and Minimal Martingale Measure may not always correctly estimates this measure. Our proposed method estimate the measure Q from the market prices of contracts available using optimal control theory without any assumption and shows high accuracy in pricing.

New Closed-form Pricing Method Pricing path dependent option (e.g. lookback option) under Lévy processes is a mathematical and computational challenge. Several papers proposed Laplace transform based approach, Wiener-Hopf factorization method, Monte Carlo technique, fast Gaussian transform method etc. But in all the analytical approaches, closed-form expression for pricing are not possible to derive with the jump process. The numerical and simulation based approaches are also quite involved and computationally expensive.

We have shown that it is possible to derive a PIDE and a closed-form arbitrage-free pricing formula for exotic option under the exponential Lévy process. The formula is simple, easy to compute and can work for any class of Lévy process avoiding complicated numerical methods in existing in the literature. The proposed method and the

Monte Carlo simulation method are very close in terms of accuracy. We hope that our method will be a satisfactory theoretical development for financial theorists and prove useful for the practitioners.

6.2 Future research

The main focus of our work was on pricing exotic options such as Asian, Barrier and Lookback for the equity options where the underlying asset is stock. Apart from the equity, there are markets where same type of exotic options are traded and underlying asset may be energy, commodities, currency etc and the price of these assets exhibit certain dynamics such as mean-reversion, changing volatility and jumps. [93] described a two factor models, one for capturing short term mean reversion and the other one for capturing long term effects with jumps

$$\begin{aligned}
\ln S_t &= f(t) + Y_t^{(1)} + Y_t^{(2)} \\
dY_t^{(1)} &= -k_1 Y_t^{(1)} dt + \sigma_1 dW_t^{(1)} \\
dY_t^{(2)} &= -k_2 Y_t^{(2)} dt + \sigma_2 dW_t^{(2)} + dZ_t \\
&< dW_t^{(1)}, dW_t^{(2)} >= \rho dt
\end{aligned}$$

where Z_t is a Lévy process described as

$$dZ_t = \mu dt + \sigma dW_t + \int_{\mathbb{R}} x \tilde{N}(dt, dx)$$

It would be quite interesting to develop a pricing and hedging method for exotic contracts with the above dynamics.
Unlike energy and commodities, the currency market exhibits changing volatility over time and it is discussed in [7]. The presence of jumps are also observed and discussed in [8]. The popular model capturing changing volatility and jumps is called Barndorff-Nielsen and Shephard (BN-S model) driven by a background Lévy process (BDLP) and it is defined by

$$\begin{aligned}
S_t &= S_0 \exp(X_t) \\
dX_t &= \mu + (\beta\sigma^2)dt + \sigma_t dW_t + \rho dZ_{\lambda t}, \\
d\sigma_t^2 &= -\lambda\sigma_t^2 dt + dZ_{\lambda t}
\end{aligned}$$

where $\mu, \beta, \rho, \lambda \in \mathbb{R}$ with $\lambda > 0$ and $\rho \leq 0$. W_t is Brownian motion and $dZ_{\lambda t}$ is subordinator. It would be also interesting to develop method for finding risk neural measure, pricing expression (specially closed-form) and hedging strategy for options in such a case.

Appendix A

Apendix of Numerical Methods

A.1 Computing $I(\eta)$ by Clenshaw Curtis quadrature rule

In this section, we will use Clenshow-Curtis rule for integration (see [40]) to calculate the integral $I(\eta)$ because of its high accuracy level and low computational time. According to Clenshow-Curtis rule for integration, any integral in [-1 1] can be written with the help of interpolation polynomial $L_n(x)$ as

$$I_n(f) = \int_{-1}^{1} f(x)dx = \int_{-1}^{1} L_n(x)dx = \int_{-1}^{1} \sum_{k=0}^{N} c_k T_k(x)dx = \sum_{k=0}^{N} c_k \mu_k, \qquad \text{(A.1.1)}$$

where $\mu_k = \int_{-1}^{1} T_k(x)dx$ are the moments of the Chaebyshev polynomials, $c_k = \frac{2}{N}\sum_{j=0}^{N} f(x_j)\cos\left(\frac{kj\pi}{N}\right)$ which is the real part of an FFT, and $x_j = \cos(j\pi/N)$.
The μ_k can be written,

$$\mu_k = \int_{-1}^{1} T_k(x)dx = \begin{cases} 0 & \text{if } k \text{ odd} \\ 2/(1-k^2) & \text{if } k \text{ even.} \end{cases}$$

A fast and accurate algorithm for computing the weights c_k in the Fejér and Clenshaw-Curtis rules in $O(N \log N)$ computation has been given by [57]. The weights are obtained as the inverse FFT of certain vectors given by explicit rational expression. Now since the above method demands the integration of a function define on $[-1, 1]$, we will convert any integration from interval $[a, b]$ to $[-1, 1]$ as follows

$$\int_a^b f(x)dx = \frac{b-a}{2} \int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{a+b}{2}\right) dx.$$

A.2 Numerical Inverse Fourier Transform

A.2.1 Properties

The Fourier transform with respect to z of function $\phi(z)$ defined on $(-\infty,\infty)$ such that $\int_{-\infty}^{\infty} \phi(z)dz < \infty$ or we say that the function is absolutely integrable, is defined as for

$$\mathcal{F}\{\phi(z)\} = \Phi(s) = \int_{-\infty}^{\infty} e^{-isz}\phi(z)dz, \quad s \in \mathbb{R},$$

where its inverse is

$$\mathcal{F}^{-1}\{\Phi(s)\} = \phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isz}\Phi(s)ds, \quad c > 0, \quad z \in \mathbb{R}.$$

The set of absolutely integrable functions has the disadvantage that it is not easy to describe all of the corresponding transforms $\Phi(s)$. The function space $\mathbb{L}^2(-\infty,\infty)$, consisting of all complex valued, measurable functions $\phi(z)$ defined on $(-\infty,\infty)$, which are "square integrable", i.e.,

$$\int_{-\infty}^{\infty} |\phi(z)|^2 dz < \infty,$$

is a more natural mathematical setting for the Fourier transform.Now, we have some interesting properties of Fourier transform on scaling and derivaties of first and second order available as follows (See [37]),

$$\mathcal{F}\{\phi(z-a)\} = e^{isa}\Phi(s),$$

$$\mathcal{F}\{\frac{\partial\phi(z)}{\partial z}\} = is\Phi(s),$$

$$\mathcal{F}\{\frac{\partial^2\phi(z)}{\partial z^2}\} = (is)^2\Phi(s).$$

We will use the above properties while applying Fourier transform on PIDE in the following sections.

A.2.2 Inversion using FFT

The integral in (5.3.4) with a semi-infinite integration interval is evaluated by numerical approximation using the trapezoidal rule and FFT. We start with the choice on the number of intervals N and the stepwidth $\triangle\eta$. A numerical approximation for $P(t,S,A)$ is given by

$$P(t,S,A) \approx \frac{e^{-k\log(S)}}{\pi}\sum_{j=1}^{L} e^{i\eta_j \log(S)}\psi(\eta_j)\triangle\eta, \tag{A.2.1}$$

where $\eta_j = (j-1)\triangle\eta, j = 1,\ldots,L$ The semi-infinite integration domain $[0,\infty)$ in the integral in (5.3.4) is approximated by a finite integration domain, where the upper limit for η in the numerical integration is $L\triangle\eta$ with some truncation error. Also, the Fourier variable η is now sampled at discrete points instead of continuous sampling. Discussion on the controls on various forms of errors in the numerical approximation procedures can be found in [71].

Recall that the FFT is an efficient numerical algorithm that computes the sum

$$y(k) = \sum_{j=1}^{L} e^{i\frac{2\pi}{L}}(j-1)(k-1)x(j), k = 1,\ldots,L. \tag{A.2.2}$$

In the current context, we would like to compute around-the-money call option prices with z taking discrete values: $-b+(m-1)\triangle z, m = 1,\ldots,L$. To effect the FFT calculations, we note that it is necessary to choose $\triangle\eta$ and $\triangle z$ such that

$$\triangle\eta\triangle z = \frac{2\pi}{L}. \tag{A.2.3}$$

For fixed L, the choice of a finer grid $\triangle\eta$ in numerical integration leads to a larger spacing $\triangle z$ on the log stock price.

A.3 Numerical Inverse of Mellin Transform

A.3.1 Properties

The Mellin transform of real valued function $\phi(z)$ defined on (0,∞) where Mellin transform with respect to s which is a real number, is definded as

$$\mathcal{M}\{\phi(z)\} = \Phi(s) = \int_0^\infty z^{s-1}\phi(z)dz, \quad s\in\mathbb{R}$$

where its inverse is

$$\mathcal{M}^{-1}\{\Phi(s)\} = \phi(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-s}\Phi(s)ds, \quad c>0$$

There are some interesting properties of Mellin Transform on scaling and derivatives of first and second order available as follows(See [37]),

$$\mathcal{M}\{\phi(az)\} = a^{-s}\Phi(s)$$

$$\mathcal{M}\{z\frac{\partial\phi(z)}{\partial z}\} = -s\Phi(s)$$

$$\mathcal{M}\{z^2\frac{\partial^2\phi(z)}{\partial z^2}\} = (-1)^2 s(s+1)\Phi(s)$$

A.3.2 Numerical Inversion

The Mellin transform is defined by the formulae [[38]]:

$$\Phi(s) = \int_0^\infty z^{s-1}\phi(z)dz, \quad Re(s) > 0 \tag{A.3.1}$$

and its inverse is

$$\phi(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-s}\Phi(s)ds, \quad c > 0$$

where one-to-one correspondence is denoted as follows, if the inverse $\Phi(s)$ function exists:

$$\phi(z) \leftrightarrow \Phi(s) \text{ or } \Phi(s) = \mathcal{M}\{\phi(z)\}.$$

The numerical Mellin inverse is first presented by [90] and later by [55].We have followed the approach proposed by [90] and can write the numerical inverse of Mellin as,

$$\phi(t) \simeq \sum_{s=1}^{N} c_s e^{-\frac{t}{2}} L_{s-1}\left(\frac{t}{2}\right) \tag{A.3.2}$$

Where

$$c_s = \sum_{n=1}^{s}(-1)^{n-1}\binom{s-1}{n-1}H_n, \quad s = 1(1)N. \tag{A.3.3}$$

and

$$H_s \equiv H(s) \equiv \frac{\Phi(s)}{2^s\Gamma(s)} \tag{A.3.4}$$

Now, we have observed that H_s is defined in integer domain and so $\Phi(s)$.But, in real case it is quite likely that the Mellin transform $\Phi(s^*) = \mathcal{M}\{f(t)\}$ will have a strip of existence for $s^* \in (a^*, b^*)$ where s^* is not an integer rather real.In such case, we will apply a linear transform as to keep H_s defined in integer domain as follows,

$$s^* = As + B, \quad s \in [1, N] \tag{A.3.5}$$

with

$$A = \frac{b^* - a^*}{N-1}, \quad B = \frac{a^*N - b^*}{N-1} \text{which maps the interval } [1, N] \text{ onto } [a^*, b^*]$$

Since the function exists in interval $[a^*, b^*]$ we can invert $\Phi(As + B)$ to recover the function $g(t)$ with the following

$$\mathcal{M}\{g(t)\} = G(s) \equiv \Phi(s^*) = \Phi(As + B), \quad s \in [1, N] \tag{A.3.6}$$

and thereafter original function $f(t) = \mathcal{M}^{-1}\Phi(s)$ can be derived by the following transformation :

$$f(t) = A\frac{g(t^A)}{t^B}$$

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