Digital Signal Processing

M. K. Naskar

Simple Digital Filters

- Lowpass
- Highpass
- Bandpass
- Bandstop
- Allpass

Pole-zero placement method

(FIR) Lowpass Filter

FIR filter

----> All zero filter

----> We have to think in terms of zeros.

----> Poles will be at trivial position only.

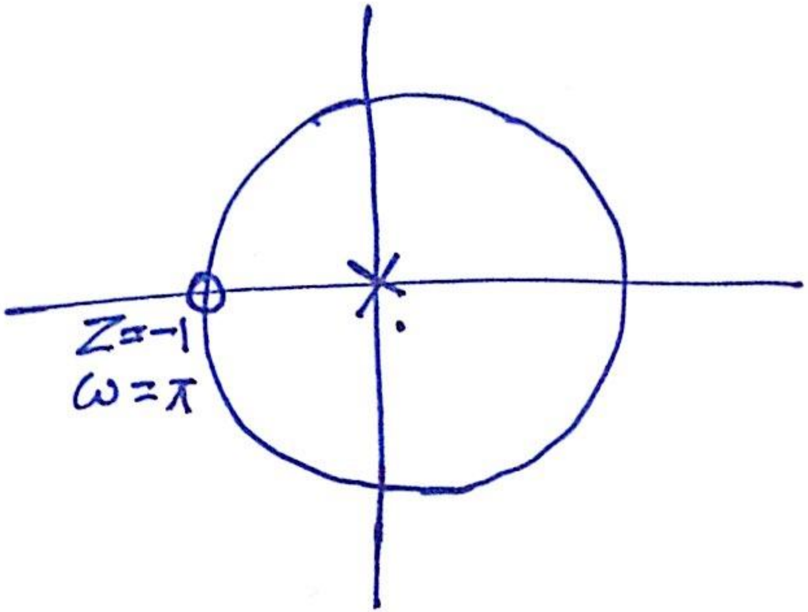
Lowpass filter

----> Low frequencies will be passed

----> High frequencies will be attenuated

----> A zero should be placed at the highest frequency

Lowpass FIR Filter :

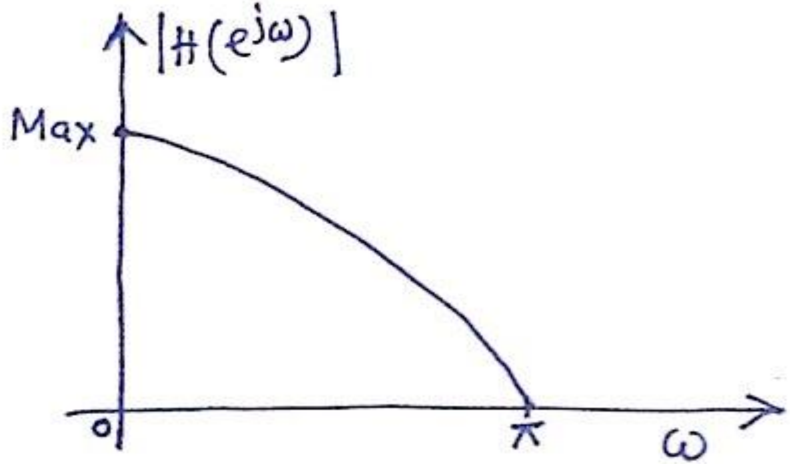


(FIR) Lowpass Filter

System function:

$$H(z) = K \cdot (1 + z^{-1})$$

Sketch of the magnitude response



Normalization of K

$$H(z)\Big|_{z=1} = 1$$

$$\Rightarrow K \cdot (1 + 1^{-1}) = 1$$

$$\Rightarrow K \cdot 2 = 1$$

$$\Rightarrow K = 1/2$$

Simplest possible FIR Lowpass Filter

System function:

$$H(z) = \frac{1}{2}\left(1 + z^{-1}\right) \longrightarrow \text{2-poin}$$

Cutoff frequency

$$H(z) = \frac{1}{2}\left(1+z^{-1}\right) \rightarrow \text{2-point moving average filter.}$$

$$\Rightarrow H(e^{j\omega}) = \frac{1}{2}\left(1+e^{-j\omega}\right)$$

$$= \frac{1}{2}\cdot e^{-j\omega/2}\left(e^{j\omega/2}+e^{-j\omega/2}\right)$$

$$= \frac{1}{2}\cdot e^{-j\omega/2}\cdot 2\cos\omega/2$$

$$= \boxed{e^{-j\omega/2}\cos(\omega/2)}$$

$$|H(e^{j\omega})|^2 = \cos^2\omega/2$$

$$\Rightarrow |H(e^{j\omega_c})|^2 = \cos^2(\omega_c/2) = \left(\frac{1}{\sqrt{2}}\right)^2 = \cancel{\tfrac{1}{2}}$$

ω_c ↑ 3 dB cut-off frequency.

$$\Rightarrow \left(\frac{1}{\sqrt{2}}\right)^2 = \cos^2\omega_c/2$$

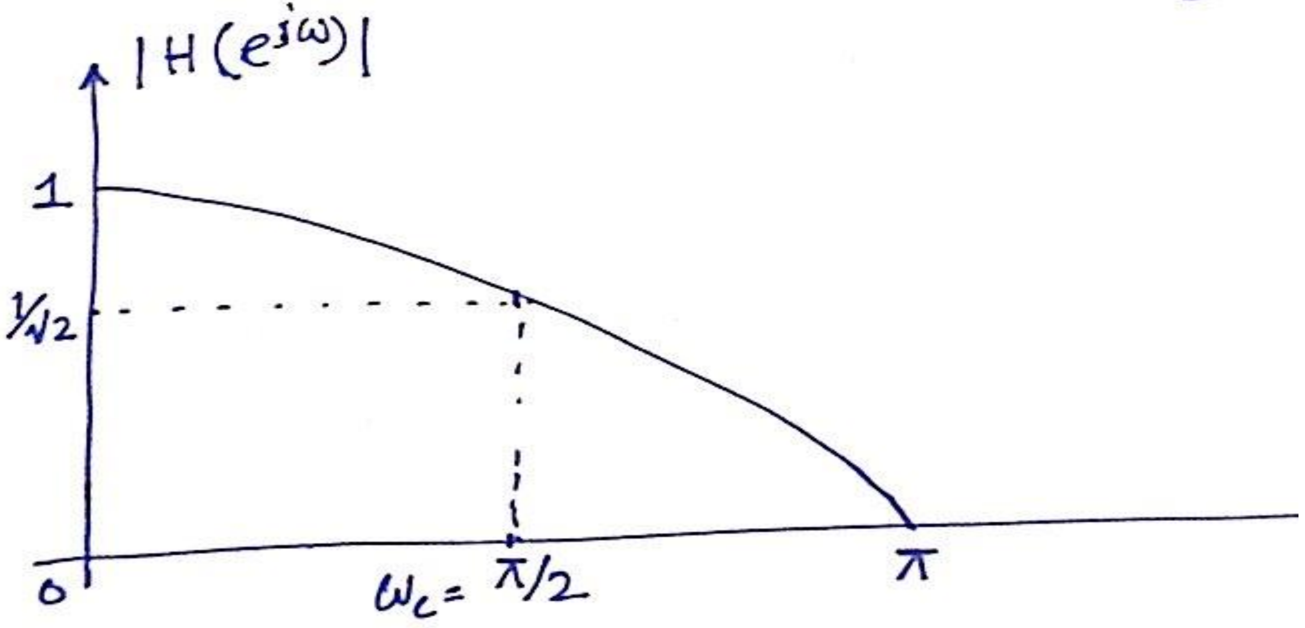
$$\Rightarrow \cos(\omega_c/2) = \cos\pi/4$$

$$\Rightarrow \omega_c/2 = \pi/4$$

$$\Rightarrow \boxed{\omega_c = \pi/2}$$

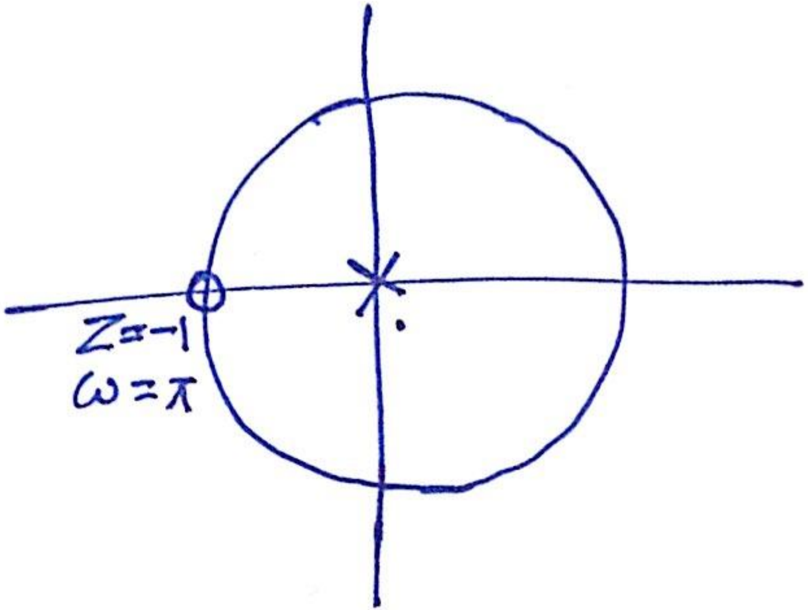
No control over the cut-off freq

Magnitude response



Magnitude response of a two-p

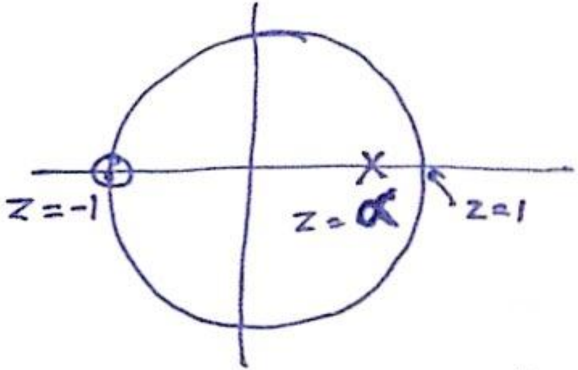
Lowpass FIR Filter :



(IIR) Lowpass Filter

system function :

$$H_2(z) = K \cdot \frac{1 + z^{-1}}{1 - \alpha z^{-1}}$$



Normalization of K

$$H(z) = K \cdot \frac{(1+z^{-1})}{1-\alpha \cdot z^{-1}}$$

Normalization

dc gain to be 1

$$\Rightarrow \quad H(z)\Big|_{z=1} = 1$$

$$\Rightarrow \quad K \cdot \frac{1+1}{1-\alpha} = 1$$

$$\Rightarrow \quad K = \left(\frac{1-\alpha}{2}\right)$$

$$\therefore \quad H(z) = \left(\frac{1-\alpha}{2}\right) \cdot \frac{(1+z^{-1})}{(1-\alpha \cdot z^{-1})}$$

Cutoff frequency

$$H(e^{j\omega}) = \left(\frac{1-\alpha}{2}\right) \cdot \frac{\cancel{(1+(e^{j\omega})^{-1})}}{(1-\alpha (e^{j\omega})^{-1})} \quad \frac{(1+e^{-j\omega})}{(1-\alpha \cdot e^{-j\omega})}$$

$$H(e^{-j\omega}) = \left(\frac{1-\alpha}{2}\right) \cdot \frac{(1+e^{j\omega})}{(1-\alpha e^{j\omega})}$$

$$\because \left|H(e^{j\omega})\right|^2 = H(e^{j\omega}) \cdot H(e^{-j\omega})$$

$$= \left(\frac{1-\alpha}{2}\right)^2 \cdot \frac{(1+e^{-j\omega})(1+e^{j\omega})}{(1-\alpha e^{-j\omega})(1-\alpha e^{j\omega})}$$

$$= \left(\frac{1-\alpha}{2}\right)^2 \cdot \frac{(1+e^{j\omega}+e^{-j\omega}+1)}{(1-\alpha e^{j\omega}-\alpha e^{-j\omega}+\alpha^2)}$$

$$\Rightarrow \left(\frac{1}{\sqrt{2}}\right)^2 = \left(\frac{1-\alpha}{2}\right)^2 \cdot \frac{(2+2\cos\omega_c)}{1-2\alpha\cos\omega_c+\alpha^2}$$

Cutoff frequency

$$\Rightarrow \frac{2 + 2\cos\omega_c}{1 - 2\alpha\cos\omega_c + \alpha^2} = \frac{2}{1 - 2\alpha + \alpha^2}$$

$$\Rightarrow \frac{1 + \cos\omega_c}{1 - 2\alpha\cos\omega_c + \alpha^2} = \frac{1}{1 - 2\alpha + \alpha^2}$$

$$\Rightarrow \cancel{1} - 2\alpha + \cancel{\alpha^2} + \cos\omega_c - \cancel{2\alpha\cos\omega_c} + \alpha^2\cos\omega_c$$

$$= \cancel{1} - \cancel{2\alpha\cos\omega_c} + \cancel{\alpha^2} = 0$$

$$\Rightarrow (\cos\omega_c)\cdot\alpha^2 - 2\cdot\alpha + \cos\omega_c = 0 \quad \Rightarrow \boxed{\cos\omega_c = \frac{2\alpha}{1+\alpha^2}}$$

Pole position

$\Rightarrow$ $(\cos\omega_c)\cdot\alpha^2 - 2\cdot\alpha + \cos\omega_c = 0 \quad \Rightarrow \quad \boxed{\cos\omega_c = \frac{2\alpha}{1+\alpha^2}}$

$\Rightarrow$ $\alpha^2 - 2\cdot\alpha\cdot\frac{1}{\cos\omega_c} + 1 = 0$

$\Rightarrow$ $\alpha^2 - 2\cdot\alpha\cdot\left(\frac{1}{\cos\omega_c}\right) + \left(\frac{1}{\cos\omega_c}\right)^2 - \left(\frac{1}{\cos\omega_c}\right)^2 + 1 = 0$

$\Rightarrow$ $(\alpha - \sec\omega_c)^2 - \sec^2\omega_c + 1 = 0$

$\Rightarrow$ $(\alpha - \sec\omega_c)^2 - \cancel{1} - \tan^2\omega_c + \cancel{1} = 0$

$\Rightarrow$ $(\alpha - \sec\omega_c - \tan\omega_c)(\alpha - \sec\omega_c + \tan\omega_c) = 0$

$\Rightarrow$ $\alpha - \sec\omega_c = \tan\omega_c$ / $\alpha - \sec\omega_c = -\tan\omega_c = \frac{1}{\cos\omega_c} - \frac{\sin\omega_c}{\cos\omega_c} = \frac{1-\sin\omega_c}{\cos\omega_c}$

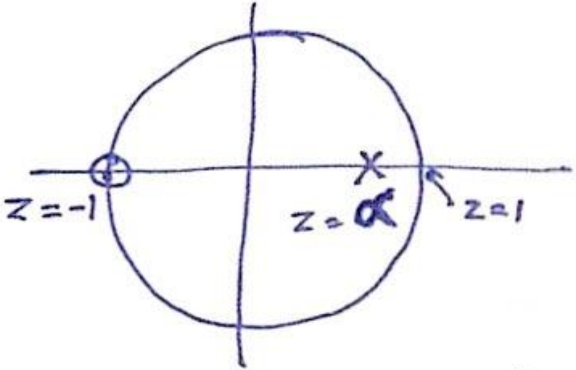
$\Rightarrow$ $\alpha = \sec\omega_c + \tan\omega_c = \frac{1}{\cos\omega_c} + \frac{\sin\omega_c}{\cos\omega_c} = \left(\frac{1+\sin\omega_c}{\cos\omega_c}\right)$

$\boxed{\alpha = \frac{1-\sin\omega_c}{\cos\omega_c}}$ Correct soln. as $|\alpha| < 1$ for stability.

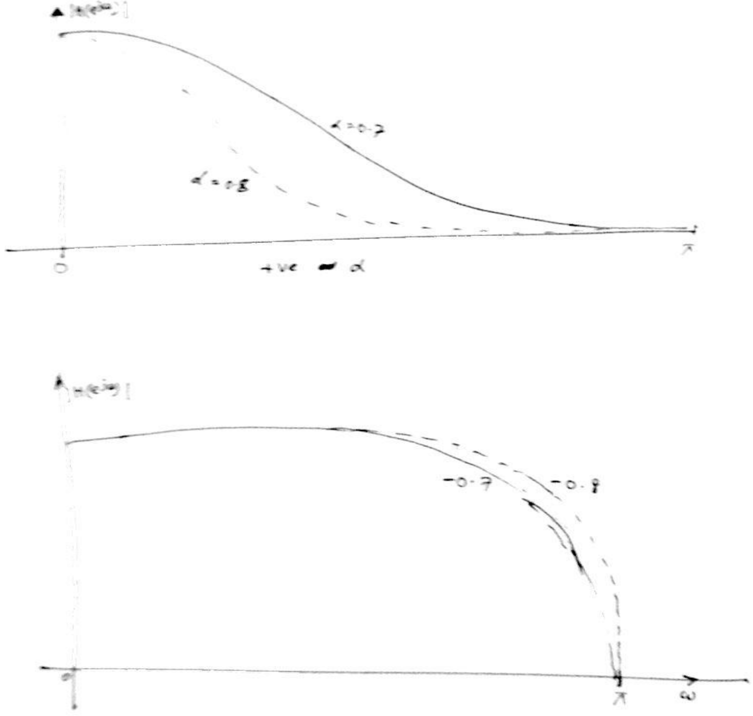
(IIR) Lowpass Filter

system function :

$$H_2(z) = K \cdot \frac{1 + z^{-1}}{1 - \alpha z^{-1}}$$



Magnitude response



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(FIR) Highpass Filter

FIR filter

----> All zero filter

----> We have to think in terms of zeros.

----> Poles will be at trivial position only.

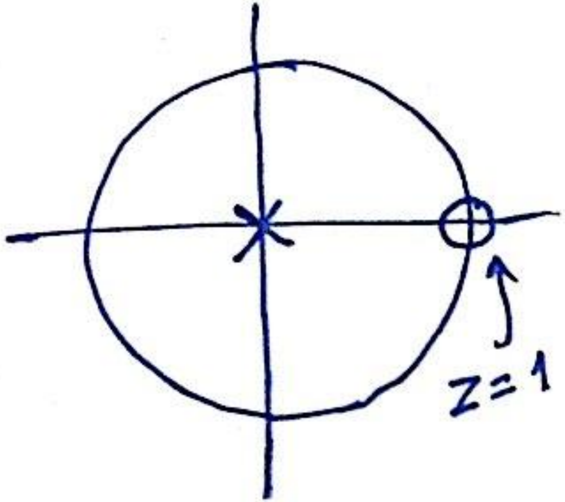
Highpass filter

----> High frequencies will be passed

----> Low frequencies will be attenuated

----> A zero should be placed at the lowest frequency

Highpass (FIR) Fil



- H
- Po

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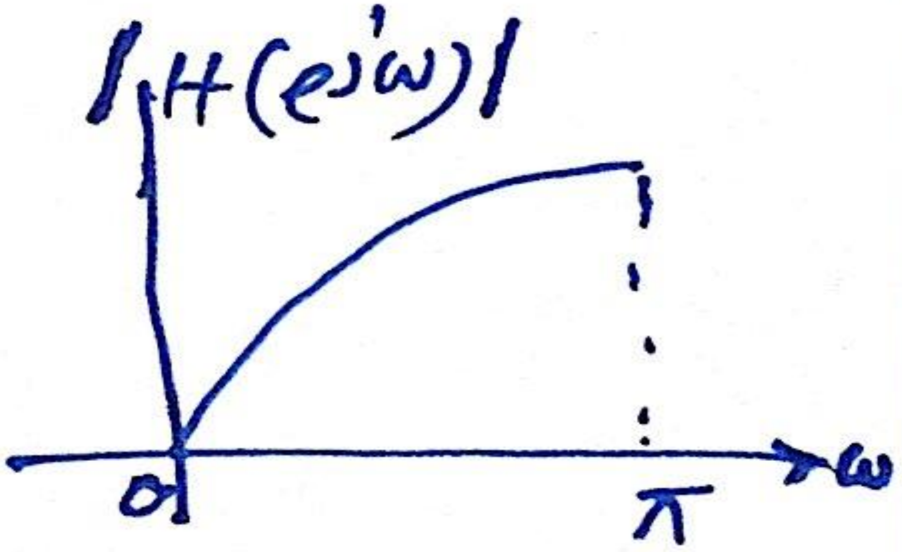
(FIR) Highpass Filter

System function:

$$H(z) = K \cdot (1 - z^{-1})$$

Sketch of the magnitude response



Normalization of K

Max. value of $H(z)$ occurs at $\omega = \pi$ or $z = -1$

$$H(z)\Big|_{z=-1} = K \cdot \left(1 - \frac{1}{(-1)}\right) = K \cdot 2$$

we normalize it at $z=-1$,

or, $\omega = \pi$

$$1 = K \cdot 2$$

$$\Rightarrow K = 1/2$$

$$\therefore \quad H(z) = \frac{1}{2}\left(1 - z^{-1}\right)$$

Cutoff frequency

$$\therefore \quad H(z) = \frac{1}{2}(1 - z^{-1})$$

$$H(e^{j\omega}) = \frac{1}{2}(1 - e^{-j\omega})$$

$$= \frac{1}{2} e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})$$

$$= \frac{1}{2} \cdot e^{-j\omega/2} \cdot 2j.\sin(\omega/2)$$

$$= j.\sin(\omega/2) \cdot e^{-j\omega/2}$$

$$|H(e^{j\omega_c})|^2 = \sin^2(\omega_c/2) = \left(\frac{1}{\sqrt{2}}\right)^2$$

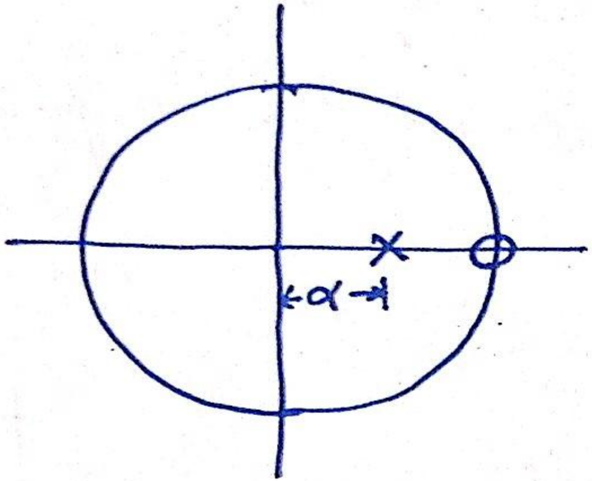
$$\Rightarrow \quad \sin \omega_c/2 = \sin \pi/4$$

$$\Rightarrow \quad \omega_c/2 = \pi/4$$

$$\Rightarrow \quad \boxed{\omega_c = \pi/2}$$

No control over the cut-off frequency.

(IIR) Highpass Filter



$$H(z) = K \cdot \frac{1 - z^{-1}}{1 - \alpha z^{-1}}$$

Normalization of K

$$\left|H(z)\right|_{z=-1} = 1$$

$$\Rightarrow \quad K = \left(\frac{1+\alpha}{2}\right)$$

$$\therefore \quad H(z) = \left(\frac{1+\alpha}{2}\right) \cdot \frac{1-z^{-1}}{1-\alpha z^{-1}}$$

Cutoff frequency

$$H(z) = \left(\frac{1+\alpha}{2}\right) \cdot \frac{(1 - z^{-1})}{(1 - \alpha z^{-1})}$$

$$\Rightarrow \quad H(e^{j\omega}) = \left(\frac{1+\alpha}{2}\right) \cdot \left(\frac{1 - e^{-j\omega}}{1 - \alpha \cdot e^{-j\omega}}\right)$$

$$\Rightarrow \quad H(e^{-j\omega}) = \left(\frac{1+\alpha}{2}\right) \left(\frac{1 - e^{j\omega}}{1 - \alpha \cdot e^{j\omega}}\right)$$

We have, $|H(e^{j\omega})|^2 = H(e^{j\omega}) \cdot H(e^{-j\omega})$

$$= \left(\frac{1+\alpha}{2}\right)^2 \frac{(1 - e^{-j\omega})(1 + e^{j\omega})}{(1 - \alpha \cdot e^{-j\omega})(1 - \alpha \cdot e^{j\omega})}$$

At cut-off,

$$\left(\frac{1}{\sqrt{2}}\right)^2 = \left(\frac{1+\alpha}{2}\right)^2 \cdot \frac{(1 - 2\cos\omega_c + 1)}{(1 - 2\alpha\cos\omega_c + \alpha^2)}$$

$$\Rightarrow \quad \frac{1}{2} = \left(\frac{1+\alpha}{2}\right)^2 \cdot \frac{2(1 - \cos\omega_c)}{1 - 2\alpha\cos\omega_c + \alpha^2}$$

$$\Rightarrow \quad 1 - 2\alpha \cdot \cos\omega_c + \alpha^2 = (1+\alpha)^2 - (1+\alpha)^2 \cdot \cos\omega_c$$

$$\Rightarrow \quad (1+\alpha)^2 \cdot \cos\omega_c - 2\alpha \cdot \cos\omega_c = (1+\alpha)^2 - 1 - \alpha^2$$

$$\Rightarrow \quad \cos\omega_c \left[1 + 2\alpha + \alpha^2 - 2\alpha\right] = (1 + \alpha^2 + 2\alpha - 1 - \alpha^2)$$

$$\Rightarrow \quad \cos\omega_c \cdot (1 + \alpha^2) = 2\alpha.$$

$$\Rightarrow \quad \boxed{\cos\omega_c = \frac{2\alpha}{1+\alpha^2}}$$

Pole position

$$\cos\omega_c = \frac{2\alpha}{1+\alpha^2}$$

$$\Rightarrow \cos\omega_c \cdot \alpha^2 + \cos\omega_c = 2\alpha$$

$$\Rightarrow \cos\omega_c \cdot \alpha^2 - 2\cdot\alpha + \cos\omega_c = 0$$

$$\Rightarrow \alpha^2 - \frac{2}{\cos\omega_c}\cdot\alpha + 1 = 0$$

$$\Rightarrow \alpha^2 - 2\cdot\alpha\cdot\frac{1}{\cos\omega_c} + \left(\frac{1}{\cos\omega_c}\right)^2 + 1 - \left(\frac{1}{\cos\omega_c}\right)^2 = 0$$

$$\Rightarrow \left(\alpha - \frac{1}{\cos\omega_c}\right)^2 + \left(1 - \sec^2\omega_c\right) = 0$$

$$\Rightarrow \left(\alpha - \frac{1}{\cos\omega_c}\right)^2 - \tan^2\omega_c = 0$$

$$\Rightarrow \left(\alpha - \frac{1}{\cos\omega_c}\right)^2 - \left(\frac{\sin\omega_c}{\cos\omega_c}\right)^2 = 0$$

$$\Rightarrow \left(\alpha - \frac{1}{\cos\omega_c} + \frac{\sin\omega_c}{\cos\omega_c}\right)\left(\alpha - \frac{1}{\cos\omega_c} - \frac{\sin\omega_c}{\cos\omega_c}\right) = 0$$

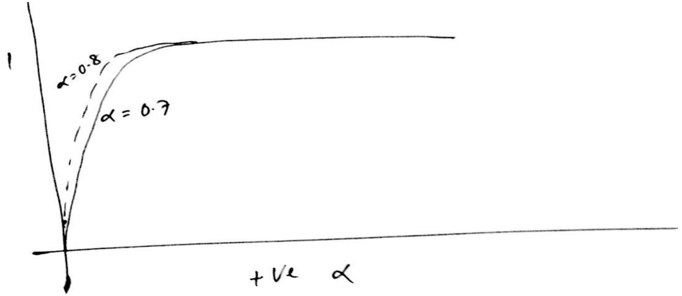
So, $\alpha = \frac{1}{\cos\omega_c} - \frac{\sin\omega_c}{\cos\omega_c}$ or, $\alpha = \frac{1}{\cos\omega_c} + \frac{\sin\omega_c}{\cos\omega_c}$

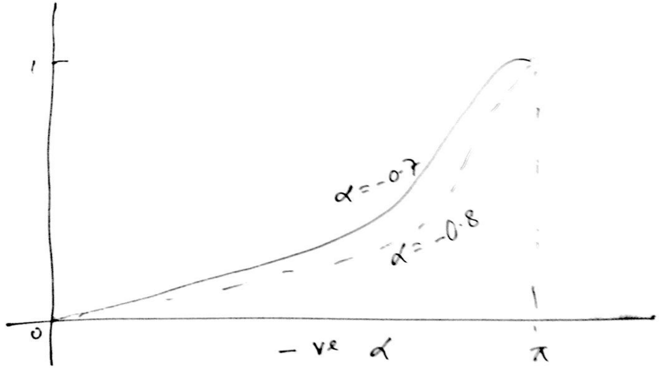
$= \frac{1-\sin\omega_c}{\cos\omega_c}$ $\qquad$ $= \frac{1+\sin\omega_c}{\cos\omega_c}$

For stability, $|\alpha| < 1$

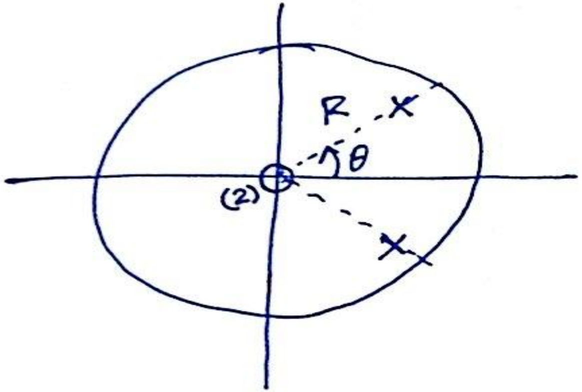
Hence, $\boxed{\alpha = \frac{1-\sin\omega_c}{\cos\omega_c}}$

Magnitude response



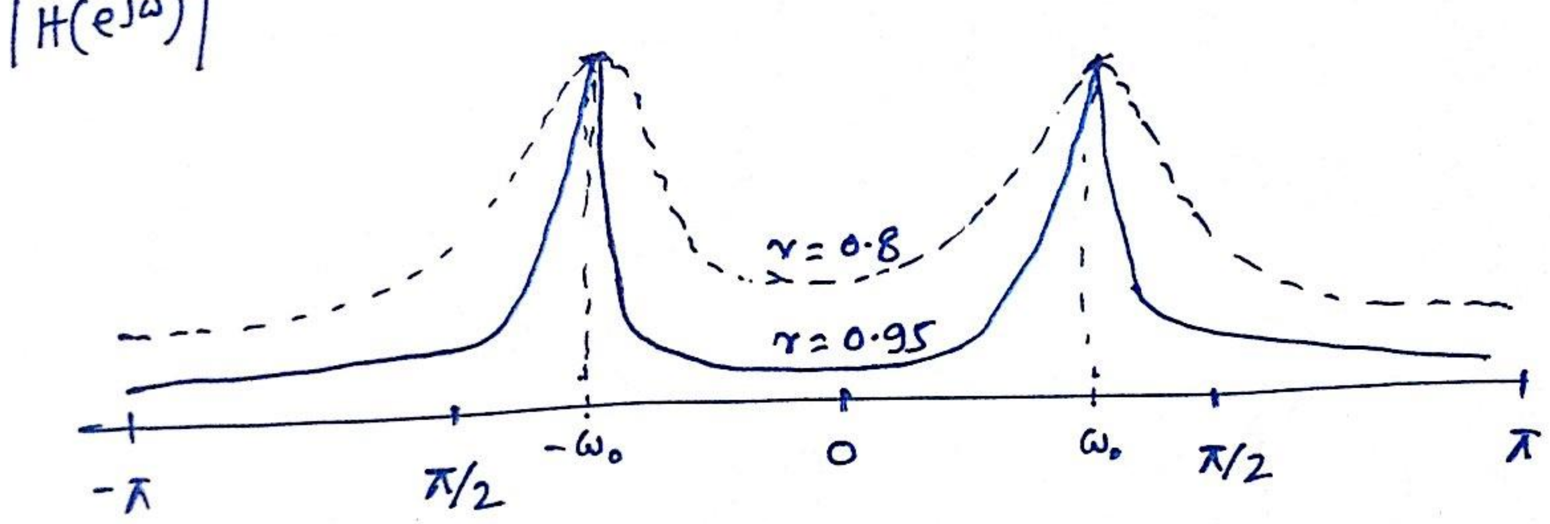


Resonators (Resons)



$$H(z) = K \cdot \frac{1}{(1 - Re^{j\theta} \cdot z^{-1})(1 - R \cdot e^{-j\theta} z^{-1})}$$

$$= K \cdot \frac{1}{1 - (2R\cos\theta) z^{-1} + R^2 z^{-2}}$$



|H(e^jω)|
r = 0.8
r = 0.95
-π
π/2
-ω₀
0
ω₀
π/2
π

Design Steps :

(a) Choose the bandwidth B and the resonant frequency ω_0.

(b) Calculate the pole radius R from the bandwidth B.

$$R = 1 - B/2$$

(c) Calculate the cosine of the pole angle θ,

$$\cos\theta = \frac{2R}{1+R^2} \cdot \cos\omega_0$$

If $R \approx 1$, $\cos\theta \approx \cos\omega_0 \Rightarrow \theta \approx \omega_0$

(d) Calculate the gain factor K,

$$K = (1-R^2)\sin\theta$$

So that, $\left|H(e^{j\omega})\right|_{\omega=\omega_0} = 1$

(e) The system function of the filter is

$$H(z) = K \times \frac{1}{(1-Re^{j\theta} \cdot z^{-1})(1-Re^{-j\theta} \cdot z^{-1})}$$

$$|H(e^{j\omega})|^2 = H(e^{j\omega}) \cdot H(e^{-j\omega})$$

$$= \frac{K^2}{\left[2R\cos\omega - (R^2+1)\cdot\cos\theta\right]^2 + (1-R^2)^2\sin^2\theta}$$

Maximum value of $|H(e^{j\omega})|$ is reached when the first term in the denominator vanishes, i.e. at $\omega = \omega_0$, where

$$2R\cos\omega_0 - (R^2+1)\cdot\cos\theta = 0$$

$$\Rightarrow \cos\omega_0 = \left(\frac{R^2+1}{2R}\right)\cdot\cos\theta$$

$$\Rightarrow \boxed{\cos\theta = \left(\frac{2R}{1+R^2}\right)\cdot\cos\omega_0}$$

$$|H(e^{j\omega})|^2_{max} = \frac{K^2}{(1-R^2)^2\sin^2\theta}$$

For normalization,

$$\boxed{K = (1-R^2)\sin\theta}$$

Bandpass filter :

$$r = 1 - \left(\frac{BW}{F_s}\right) \times \pi$$

$$\Delta\omega = 2(1-r) \quad , \quad \Delta\omega \propto (1-r)$$

$$\Rightarrow \frac{BW}{2F_s} \neq$$

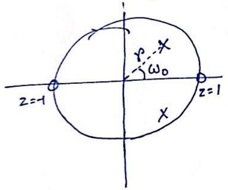
$$\Rightarrow \cancel{2}\pi\left(\frac{BW}{F_s}\right) = \cancel{2}(1-r)$$

magnitude of the poles

$$\Rightarrow \boxed{r \approx 1 - \left(\frac{BW}{F_s}\right) \times \pi}$$

angle of the pole

$$\omega_0 = 2\pi\left(\frac{f_0}{F_s}\right) \quad , \quad f_0 \text{ is the center frequency.}$$

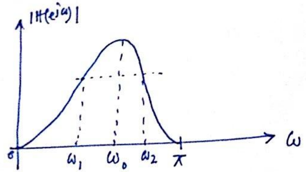


Given :

i) Center freq^n f_0 Hz

ii) 3-dB BW in Hz

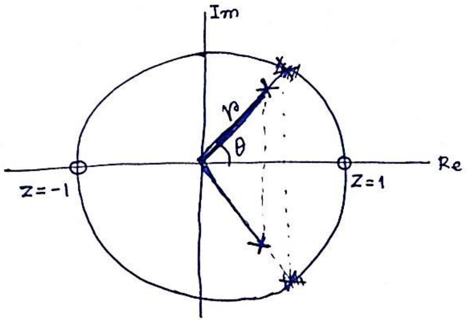
$$H(z) = K \cdot \frac{(z+1)(z-1)}{(z - re^{j\omega_0})(z - re^{-j\omega_0})} = \frac{K(z^2 - 1)}{z^2 - 2r\cos\omega_0 \cdot z + r^2}$$



$$\Delta\omega = \omega_2 - \omega_1$$

$$K = \frac{(1-r)\sqrt{1 - 2r\cos 2\omega_0 + r^2}}{2\,|\sin\omega_0|}$$

Bandpass Filter



Pole-zero diagram

System function:

$$H(z) = K \cdot \frac{(1-z^{-1})(1+z^{-1})}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})}$$

$$= K \cdot \frac{(1-z^{-2})}{1-2r\cos\theta \cdot z^{-1} + r^2 z^{-2}}$$

where, K is the gain constant

Normalization

K will be chosen to normalize the maximum magnitude to unity.

The frequency response is given by

$$H(e^{j\omega}) = K \cdot \frac{(1 - e^{-2j\omega})}{1 - 2r\cos\theta \cdot e^{-j\omega} + r^2 e^{-2j\omega}}$$

$$H(e^{-j\omega}) = K \cdot \frac{(1 - e^{2j\omega})}{1 - 2r\cos\theta \cdot e^{j\omega} + r^2 e^{2j\omega}}$$

$$\therefore |H(e^{j\omega})|^2 = H(e^{j\omega}) \cdot H(e^{-j\omega})$$

$$= \frac{4K^2}{\left[\frac{(1+r^2)\cos\omega - 2r\cos\theta}{\sin\omega}\right]^2 + (1-r^2)^2}$$

Maximum value of $|H(e^{j\omega})|$ is reached when the first term in the denominator vanishes, i.e. at $\omega = \omega_0$, where

$$\cos\omega_0 = \left(\frac{2r\cos\theta}{1+r^2}\right) = \beta \text{ (Let)}$$

So, $$|H(e^{j\omega})|^2_{max} = \frac{4K^2}{(1-r^2)^2}$$

$$\Rightarrow |H(e^{j\omega})|_{max} = \frac{2K}{(1-r^2)} = 1 \text{ (normalized value)}$$

$$\Rightarrow K = (1-r^2)/2$$

~~Cutoff frequencies~~ Bandwidth

Putting, $K = (1-r^2)/2$

$$|H(e^{j\omega})|^2 = \frac{1}{1 + \left[\frac{(1+r^2)\cos\omega - 2r\cos\theta}{(1-r^2)\sin\omega}\right]^2}$$

At cutoff frequencies, $|H(e^{j\omega})|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

So, $(1+r^2)\cos\omega - 2r\cos\theta = \pm(1-r^2)\sin\omega$

or, $(1+r^2)\cos\omega_1 - 2r\cos\theta = (1-r^2)\sin\omega_1$ —(i)

and $(1+r^2)\cos\omega_2 - 2r\cos\theta = -(1-r^2)\sin\omega_2$ —(ii)

Subtracting (ii) from (i) gives

$$(1+r^2)(\cos\omega_1 - \cos\omega_2) = (1-r^2)(\sin\omega_1 + \sin\omega_2)$$

$$\Rightarrow (1+r^2)\sin\frac{\omega_2+\omega_1}{2}\sin\frac{\omega_2-\omega_1}{2} = (1-r^2)\sin\frac{\omega_2+\omega_1}{2}\cdot\cos\frac{\omega_2-\omega_1}{2}$$

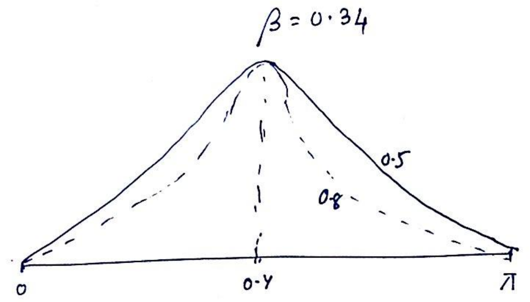
$$\Rightarrow \tan\left(\frac{\omega_2-\omega_1}{2}\right) = \frac{1-r^2}{1+r^2}$$

$$\Rightarrow \cos(\omega_2-\omega_1) = \frac{(1+r^2)^2 - (1-r^2)^2}{(1+r^2)^2 + (1-r^2)^2}$$

$$\Rightarrow \cos B = \frac{2r^2}{1+r^4} = \frac{2\alpha}{1+\alpha^2}$$ ~~Let B = Bandwidth~~ (Let, $r^2 = \alpha$)

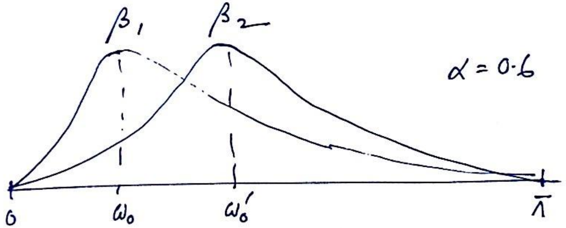
$$\Rightarrow \quad K = \left(\frac{1-\alpha}{2}\right)$$

So,
$$H(z) = \left(\frac{1-\alpha}{2}\right) \cdot \frac{1 - z^{-2}}{1 - \beta(1+\alpha) z^{-1} + \alpha z^{-2}}$$



Fixed β $\Rightarrow$ fixed centre freq., ω_0

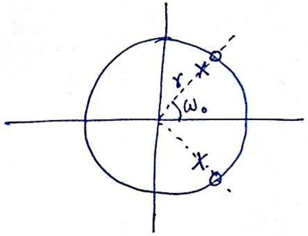
α is varied $\Rightarrow$ B is varied $\quad \alpha \uparrow \rightarrow B \downarrow$

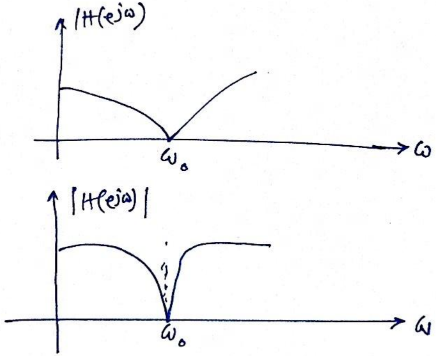


β is varied $\Rightarrow$ centre freq. is varied

α is fixed $\Rightarrow$ B is fixed.

Bandstop (Notch) Filter





$$H(z) = K \cdot \frac{(z - e^{j\omega_0})(z - e^{-j\omega_0})}{(z - re^{j\omega_0})(z - re^{-j\omega_0})} = \frac{K(z^2 - 2\cos\omega_0 z + 1)}{(z^2 - 2r\cos\omega_0 z + r^2)}$$

$$K = \frac{(1 - 2r\cos\omega_0 + r^2)}{(2 - 2\cos\omega_0)}, \quad \text{Normalization: } H(z)\big|_{z=1} = 1$$

$$r \approx 1 - (BW/F_s) \times \pi, \quad 0.9 \le r < 1$$

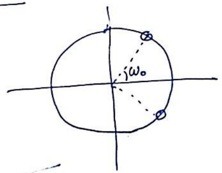
$$\omega_0 = 2\pi (F_0/F_s)$$

Center freq^n. F_0

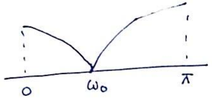
3-dB bandwidth BW

Bandstop Filter

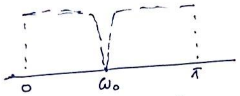
(IIR Notch)



$$1 - 2\cos\omega_0 \cdot z^{-1} + z^{-2}$$



3 dB BW is very large



We want to enchange the response near ω_0. So we introduce two more poles at the same freqn.

$$H(z) = K \cdot \frac{(1 - 2\beta z^{-1} + z^{-2})}{1 - \beta(1+\alpha) z^{-1} + \alpha z^{-2}}$$

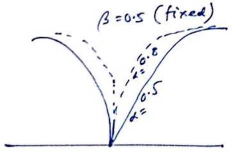
$\left.\begin{array}{l}|\alpha| < 1 \\ |\beta| < 1\end{array}\right\}$ stability.

where, $\beta = \cos\omega_0$, ω_0 is the notch freqn.

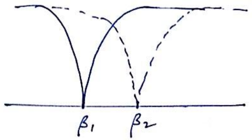
Max. value at $\omega = 0, \pi$, $\frac{2K}{1+\alpha} = 1 \Rightarrow K = \left(\frac{1+\alpha}{2}\right)$.

$$H(z) = \left(\frac{1+\alpha}{2}\right) \cdot \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1+\alpha) \cdot z^{-1} + \alpha z^{-2}}$$

$$B = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$$



α changes the bandwidth



α is fixed $\Rightarrow$ B is fixed.

All-pass Filter

An all-pass filter is defined as a system that has a constant magnitude response for all frequencies, that is,

$$|H(e^{j\omega})| = 1 \quad , \quad 0 \leq \omega \leq \pi$$

- The simplest example of an all-pass filter is a pure delay (system) of K samples. This is a trivial all-pass system that has a linear phase characteristic

$$\boxed{y(n) = x(n-K)}$$

$$\Rightarrow \quad Y(z) = z^{-K} X(z)$$

$$\Rightarrow \quad \boxed{H(z) = \frac{Y(z)}{X(z)} = z^{-K}}$$

$$\Rightarrow \quad H(e^{j\omega}) = e^{-j\omega K}$$

$$|H(e^{j\omega})| = 1 \quad , \quad \angle H(e^{j\omega}) = -\omega K$$

• A more interesting all-pass filter is described by the system function

$$H(z) = \frac{a_N + a_{N-1} z^{-1} + \cdots + a_1 z^{-N+1} + z^{-N}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

where all the filter coefficients $\{a_k\}$ are real.

$$H(z) = z^{-N} \cdot \frac{A(z^{-1})}{A(z)}$$

where $A(z) = 1 + \sum_{k=0}^{N} a_k \cdot z^{-k}$, ~~$a_0 = 1$~~

Since, $|H(e^{j\omega})|^2 = H(z)\, H(z^{-1})\Big|_{z=e^{j\omega}} = 1$

the system is an all-pass system.

$$A(z) = 1 + \sum_{k=1}^{N} a_k \cdot z^{-k}$$

$$A(z^{-1}) = 1 + \sum_{k=1}^{N} a_k \cdot z^{k}$$

$$z^{-N} A(z^{-1}) = z^{-N} + \sum_{k=1}^{N} a_k \cdot z^{k} \cdot z^{-N}$$

$$= z^{-N} + \sum_{k=1}^{N} a_k \cdot z^{-N+k}$$

$$H(z) = \frac{a_N + a_{N-1} z^{-1} + \cdots + a_1 z^{-N+1} + z^{-N}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$= \frac{z^{-N} + \sum_{k=1}^{N} a_k \cdot z^{-N+k}}{1 + \sum_{k=1}^{N} a_k \cdot z^{-k}} = z^{-N} \cdot \frac{A(z^{-1})}{A(z)}$$

where, $A(z) = 1 + \sum_{k=1}^{N} a_k \cdot z^{-k}$

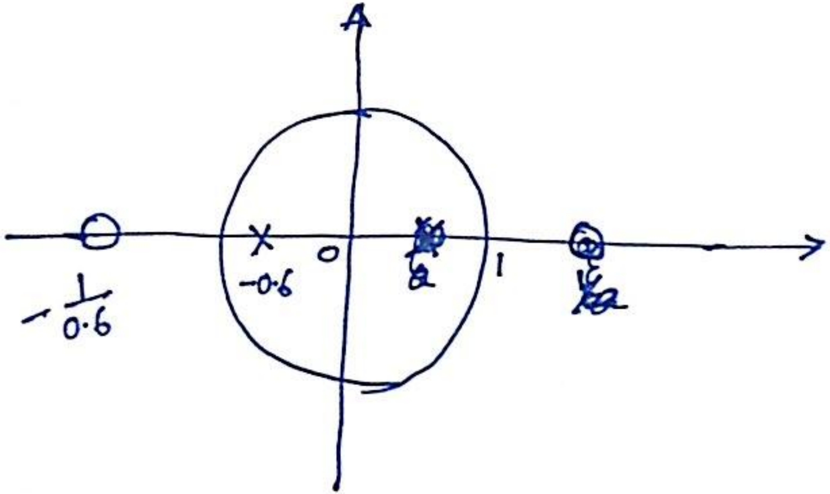
- Furthermore, if z_0 is a pole of $H(z)$, then $1/z_0$ is a zero of $H(z)$ (i.e. poles and zeros are reciprocals of one another).

- $$|H(e^{j\omega})|^2 = H(e^{j\omega}) \cdot H^*(e^{j\omega}) = H(e^{j\omega})\, H(e^{-j\omega}) = H(z)H(z^{-1})\Big|_{z=e^{j\omega}}$$
$$= e^{-j\omega N} \cdot \frac{A(e^{-j\omega})}{A(e^{j\omega})} \cdot e^{j\omega N} \cdot \frac{A(e^{j\omega})}{A(e^{-j\omega})} = 1$$

- All pass filters find applications as phase equalizers. When placed in cascade with a system that has an undesired phase response, a phase equalizer is designed to compensate for the poor phase characteristics of the system and therefore to produce an overall linear phase response.

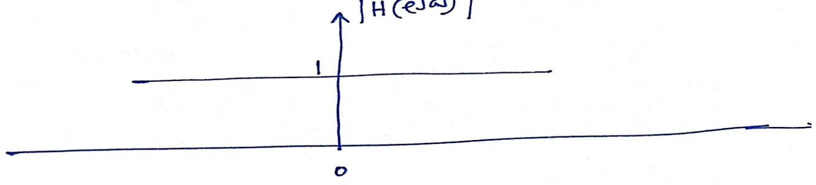
A first-order

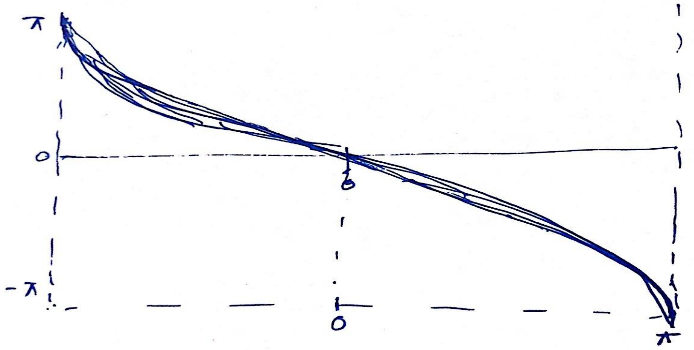
• (Single-pole single-zero) all-pass filter



$$H(z) = \frac{0.6 + z^{-1}}{1 + 0.6 z^{-1}} \qquad = \frac{0.6}{}$$

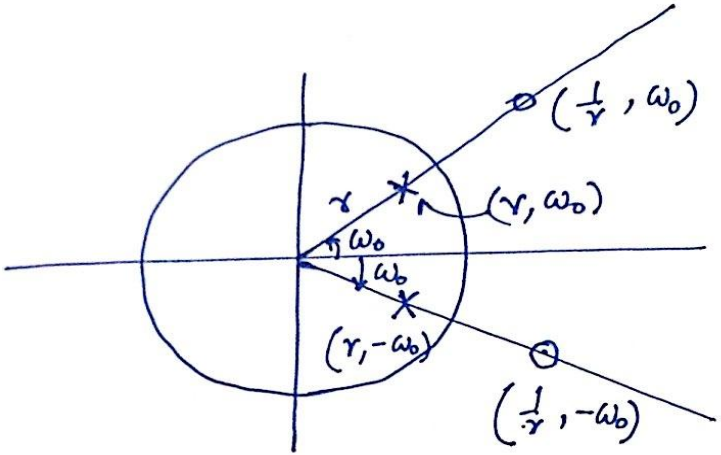
Magnitude and phase response





A second order

- (A two-pole, two-zero) all-pass filter:



$$H(z) = \frac{r^2 - 2r\cos\omega_0 z^{-1} + z^{-2}}{1 - 2r\cos\omega_0 \cdot z^{-1} + r^2 z^{-2}}$$

Magnitude and phase response